Review 1 for Exam 1.

- 6 or 7 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks, webwork.
- Exam covers:
  - Linear equations (1.1), (1.2).
  - Bernoulli equation (1.2).
  - Separable equations (1.3).
  - Homogeneous equations (1.3).
  - Exact equations (1.4).
  - Exact equations with integrating factors (1.4).
  - Modeling (1.5).

Exam overview

Remark:

- Exam problems will be: Solve this equation. We don't tell you if the equation is linear, separable, etc. You must find that out.
- If you know what type of equation is, then the equation is simple to solve.
- The difficult part in Exam 1 is to know what type of equation is the one you have to solve.

Exam overview

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\, y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)
5. Exact equations.
   (Check one equation: $N\, y' + M = 0$, and $\partial_t N = \partial_y M$.)
6. Exact equation with integrating factor.
   (Very complicated to check.)

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Example

Find every solution y to the equation $(t^2 + y^2)(t + y\, y') + 2 = 0$.

Solution: Rewrite the equation in a more standard way:

$$(t^2 + y^2) y\, y' + (t^2 + y^2) t + 2 = 0 \quad \Leftrightarrow \quad y' = -\frac{(t^2 + y^2)t + 2}{(t^2 + y^2) y}.$$

Not linear. Not Bernoulli. Not Separable. Not homogeneous.
So the equation must be exact or exact with integrating factor.

$$N = t^2 y + y^3 \quad \Rightarrow \quad \partial_t N = 2ty.$$

$$M = t^3 + t y^2 + 2 \quad \Rightarrow \quad \partial_y M = 2ty.$$

The equation is exact: $\partial_t N = \partial_y M$.

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Example

Find every solution y to the equation $(t^2 + y^2)(t + y\,y') + 2 = 0$.

Solution: $\partial_t N = \partial_y M$, $\quad \big[(t^2 + y^2)y\big]\,y' + \big[(t^2 + y^2)t + 2\big] = 0.$

There exits a potential function ψ such that

$$\partial_y \psi = N, \qquad \partial_t \psi = M.$$

$$\partial_y \psi = t^2 y + y^3 \quad \Rightarrow \quad \psi = t^2\,\frac{y^2}{2} + \frac{y^4}{4} + g(t).$$

$$t y^2 + g'(t) = \partial_t \psi = M = t^3 + t y^2 + 2.$$

$$g'(t) = t^3 + 2 \quad \Rightarrow \quad g(t) = \frac{t^4}{4} + 2t.$$

$$\psi(t, y) = \frac{1}{2}\,t^2 y^2 + \frac{y^4}{4} + \frac{t^4}{4} + 2t, \qquad \psi(t, y(t)) = c. \qquad \triangleleft$$

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Example

Find the explicit solution y to the IVP

$$y' = \frac{t(t^2 + e^t)}{4y^3}, \qquad y(0) = -\sqrt{2}.$$

Solution: Not linear. Bernoulli with $n = -3$. Numerator depends only on t, denominator depends only on y: Separable.

$$4y^3\,y' = t^3 + t e^t \quad \Rightarrow \quad \int 4y^3\,y'\,dt = \int (t^3 + t e^t)\,dt + c$$

The usual substitution: $u = y(t)$ implies $du = y'(t)\,dt$,

$$\int 4u^3\,du = \int (t^3 + t e^t)\,dt + c \quad \Rightarrow \quad u^4 = \frac{t^4}{4} + \int t e^t\,dt + c.$$

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Example

Find the explicit solution y to the IVP

$$y' = \frac{t(t^2 + e^t)}{4y^3}, \qquad y(0) = -\sqrt{2}.$$

Solution: Recall: $u^4 = \frac{t^4}{4} + \int t e^t \, dt + c$. Integration by parts:

$$\left.\begin{matrix} f = t, & g' = e^t, \\ f' = 1, & g = e^t, \end{matrix}\right\} \quad \Rightarrow \quad \int t e^t \, dt = t e^t - \int e^t \, dt = (t-1)e^t.$$

We obtain: $y^4(t) = \frac{t^4}{4} + (t-1)e^t + c$. The initial condition:

$$\left(-\sqrt{2}\right)^4 = 0 + (0-1) + c \quad \Rightarrow \quad 4 = -1 + c \quad \Rightarrow \quad c = 5.$$

We conclude: $y^4(t) = \frac{t^4}{4} + (t-1)e^t + 5$. Implicit form.

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Example

Find the explicit solution y to the IVP

$$y' = \frac{t(t^2 + e^t)}{4y^3}, \qquad y(0) = -\sqrt{2}.$$

Solution: Recall: $y^4(t) = \frac{t^4}{4} + (t-1)e^t + 5$. Implicit form.

The explicit form of the solution is one of:

$$y(t) = \pm \left[\frac{t^4}{4} + (t-1)e^t + 5\right]^{1/4}.$$

The initial condition implies $y(0) = -\sqrt{2} < 0$.

We conclude that the unique solution to the IVP is

$$y(t) = -\left[\frac{t^4}{4} + (t-1)e^t + 5\right]^{1/4}. \qquad \triangleleft$$

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Example

Find every solution y of the equation $y' = \dfrac{3y^2 - t^2}{2ty}$.

Solution: Not linear. Bernoulli $n = -1$: $y' = \dfrac{3y}{2t} - \dfrac{t}{2y}$.
Not separable. Every term on the right hand side is of the form $t^n y^m$ with $n + m = 2$. Homogeneous.

$$y' = \frac{3y^2 - t^2}{2ty}\,\frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad \Rightarrow \quad y' = \frac{3\left(\frac{y}{t}\right)^2 - 1}{2\left(\frac{y}{t}\right)}.$$

We introduce the change of unknown:

$$v = \frac{y}{t} \quad \Rightarrow \quad y = t\,v \quad \Rightarrow \quad y' = v + t\,v'.$$

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Example

Find every solution y of the equation $y' = \dfrac{3y^2 - t^2}{2ty}$.

Solution: $y' = \dfrac{3\left(\frac{y}{t}\right)^2 - 1}{2\left(\frac{y}{t}\right)}$, $\quad v = \dfrac{y}{t}$, $\quad y' = v + t\,v'$.

$$v + t\,v' = \frac{3v^2 - 1}{2v} \quad \Rightarrow \quad t\,v' = \frac{3v^2 - 1}{2v} - v = \frac{3v^2 - 1 - 2v^2}{2v}$$

$$t\,v' = \frac{v^2 - 1}{2v} \quad \Rightarrow \quad \frac{2v}{v^2 - 1}\,v' = \frac{1}{t}.$$

This is a separable equation for v: $\displaystyle\int \frac{2v}{v^2 - 1}\,v'\,dt = \int \frac{1}{t}\,dt + c.$

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Example

Find every solution y of the equation $y' = \dfrac{3y^2 - t^2}{2ty}$.

Solution: $\displaystyle\int \frac{2v}{v^2 - 1}\, v'\, dt = \int \frac{1}{t}\, dt + c.$

The substitution $u = v^2 - 1$ implies $du = 2v\, v'\, dt$. So,

$$\int \frac{du}{u} = \int \frac{1}{t}\, dt + c \quad \Rightarrow \quad \ln(|u|) = \ln(|t|) + c \quad \Rightarrow \quad |u| = c_1\, |t|.$$

where $c_1 = e^c$. Substitute back: $|v^2 - 1| = c_1\, |t|$. Finally, $v = y/t$,

$$\left| \frac{y^2}{t^2} - 1 \right| = c_1\, |t| \quad \Rightarrow \quad |y^2 - t^2| = c_1\, |t|^3.$$

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Example

A water tank initially has $V_0 = 100$ liters of water with Q_0 grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates r_i and r_o such that:

(a) The tank water volume is constant.

(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution:

Part (a): Water volume constant implies $r_i = r_o = r$. Then $V'(t) = 0$, so $V(t) = V_0$.

Part (b): First find the salt in the tank $Q(t)$: $\dfrac{dQ}{dt} = r_i q_i - r_o q_o(t)$.
Incoming fresh water: $q_i = 0$. Mixing: $q_o(t) = Q(t)/V(t)$.

$$\frac{dQ}{dt} = -\frac{r}{V_0}\, Q(t) \quad \Rightarrow \quad Q(t) = Q_0\, e^{-rt/V_0}.$$

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Example

A water tank initially has $V_0 = 100$ liters of water with Q_0 grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates r_i and r_o such that:

(a) The tank water volume is constant.

(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution: Recall: $Q(t) = Q_0\, e^{-rt/V_0}$. Condition for r:

$$Q(t_1) = \frac{Q_0}{100} \;\Rightarrow\; Q_0\, e^{(-rt_1/V_0)} = \frac{Q_0}{100} \;\Rightarrow\; -\frac{rt_1}{V_0} = \ln\Big(\frac{1}{100}\Big).$$

$$\frac{rt_1}{V_0} = \ln(100) \;\Rightarrow\; r = \frac{V_0}{t_1}\ln(100) \;\Rightarrow\; r = 4\ln(100). \qquad \triangleleft$$

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Example

Find the solution y to the IVP

$$y' = \frac{2}{t}\,y - \frac{\sin(t)}{t}\,y^2, \qquad y(2\pi) = 2\pi, \qquad t > 0.$$

Solution: Not linear. Bernoulli for $n = 2$. Divide by y^2.

$$\frac{y'}{y^2} - \frac{2}{t}\,\frac{1}{y} = -\frac{\sin(t)}{t}, \qquad v = \frac{1}{y} \quad\Rightarrow\quad v' = -\frac{y'}{y^2}.$$

$$-v' - \frac{2}{t}\,v = -\frac{\sin(t)}{t} \quad\Rightarrow\quad v' + \frac{2}{t}\,v = \frac{\sin(t)}{t}.$$

We solve the linear equation with the integrating factor method.

$$A(t) = \int \frac{2}{t}\,dt = 2\ln(t) = \ln\big(t^2\big) \quad\Rightarrow\quad \mu(t) = t^2.$$

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Example

Find the solution y to the IVP

$$y' = \frac{2}{t}\,y - \frac{\sin(t)}{t}\,y^2, \qquad y(2\pi) = 2\pi, \qquad t > 0.$$

Solution: Recall: $\mu(t) = t^2$. Then,

$$t^2\Big(v' + \frac{2}{t}\,v\Big) = t^2\,\frac{\sin(t)}{t} \quad \Rightarrow \quad \big(t^2\,v\big)' = t\sin(t).$$

Integrating: $t^2\,v = \displaystyle\int t\sin(t)\,dt + c$. The right hand side can be computed integrating by parts,

$$\int t\sin(t)\,dt = -t\cos(t) + \int \cos(t)\,dt, \quad \begin{cases} f = t, & g' = \sin(t), \\ f' = 1, & g = -\cos(t). \end{cases}$$

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Example

Find the solution y to the IVP

$$y' = \frac{2}{t}\,y - \frac{\sin(t)}{t}\,y^2, \qquad y(2\pi) = 2\pi, \qquad t > 0.$$

Solution: $\displaystyle\int t\sin(t)\,dt = -t\cos(t) + \int \cos(t)\,dt$. Then,

$$t^2\,v = -t\cos(t) + \sin(t) + c \quad \Rightarrow \quad t^2\,\frac{1}{y} = -t\cos(t) + \sin(t) + c.$$

The initial condition: $4\pi^2\,\dfrac{1}{2\pi} = -2\pi\cos(2\pi) + 0 + c$, so $c = 4\pi$.

$$y = \frac{t^2}{\sin(t) - t\cos(t) + 4\pi} \qquad \triangleleft$$

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Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

Solution: We first verify if the equation is not exact.

$$N = \left(x^3 e^y + \frac{x}{y}\right) \quad \Rightarrow \quad \partial_x N = 3x^2 e^y + \frac{1}{y}.$$

$$M = (2x^2 e^y + 1) = 0 \quad \Rightarrow \quad \partial_y M = 2x^2 e^y.$$

So the equation is not exact. We now compute

$$\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)} = \frac{-x^2 e^y - \frac{1}{y}}{x\left(x^2 e^y + \frac{1}{y}\right)} = -\frac{1}{x}.$$

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Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

Solution: Recall: $\dfrac{\partial_y M - \partial_x N}{N} = -\dfrac{1}{x}$. Therefore,

$$\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x} \quad \Rightarrow \quad \ln(\mu) = -\ln(x) = \ln\left(\frac{1}{x}\right) \quad \Rightarrow \quad \mu(x) = \frac{1}{x}.$$

So the equation $\left(x^2 e^y + \dfrac{1}{y}\right) y' + \left(2x e^y + \dfrac{1}{x}\right) = 0$ is exact. Indeed,

$$\left.\begin{aligned} \tilde{N} &= \left(x^2 e^y + \frac{1}{y}\right) \quad \Rightarrow \quad \partial_x \tilde{N} = 2x e^y, \\ \tilde{M} &= \left(2x e^y + \frac{1}{x}\right) \quad \Rightarrow \quad \partial_y \tilde{M} = 2x e^y, \end{aligned}\right\} \quad \Rightarrow \quad \partial_x \tilde{N} = \partial_y \tilde{M}.$$