Variable coefficients second order linear ODE (Sect. 2.1).

- Second order linear ODE.
- Superposition property.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel's theorem on the Wronskian.
- Special Second order nonlinear equations.

Second order linear differential equations.

Definition

Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t) \tag{1}$$

is called a *second order linear* differential equation with *variable coefficients*. The equation in (1) is called *homogeneous* iff for all $t \in \mathbb{R}$ holds

$$b(t) = 0.$$

The equation in (1) is called of *constant coefficients* iff a_1, a_0, and b are constants.

Remark: The notion of an homogeneous equation presented here is not the same as the notion presented in the previous chapter.

Second order linear differential equations.

Example

(a) A second order, linear, homogeneous, constant coefficients equation is

$$y'' + 5y' + 6 = 0.$$

(b) A second order order, linear, constant coefficients, non-homogeneous equation is

$$y'' - 3y' + y = 1.$$

(c) A second order, linear, non-homogeneous, variable coefficients equation is

$$y'' + 2t\,y' - \ln(t)\,y = e^{3t}.$$

(d) Newton's second law of motion ($ma = f$) for point particles of mass m moving in one space dimension under a force $f : \mathbb{R} \to \mathbb{R}$ is given by

$$m\,y''(t) = f(t). \qquad \triangleleft$$

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Superposition property.

Theorem
If the functions y_1 and y_2 are solutions to the homogeneous linear equation

$$y'' + a_1(t)\,y' + a_0(t)\,y = 0, \qquad (2)$$

then the linear combination $c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1, $c_2 \in \mathbb{R}$.

Proof: Verify that the function $y = c_1y_1 + c_2y_2$ satisfies Eq. (2) for every constants c_1, c_2, that is,

$$(c_1y_1 + c_2y_2)'' + a_1(t)(c_1y_1 + c_2y_2)' + a_0(t)(c_1y_1 + c_2y_2)$$

$$= (c_1y_1'' + c_2y_2'') + a_1(t)(c_1y_1' + c_2y_2') + a_0(t)(c_1y_1 + c_2y_2)$$

$$= c_1\big[y_1'' + a_1(t)y_1' + a_0(t)y_1\big] + c_2\big[y_2'' + a_1(t)y_2' + a_0(t)y_2\big] = 0.$$

$\square$

Variable coefficients second order linear ODE (Sect. 2.1).

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Existence and uniqueness of solutions.

Theorem (Variable coefficients)

If the functions $a, b : (t_1, t_2) \to \mathbb{R}$ *are continuous, the constants* $t_0 \in (t_1, t_2)$ *and* y_0, $y_1 \in \mathbb{R}$, *then there exists a unique solution* $y : (t_1, t_2) \to \mathbb{R}$ *to the initial value problem*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

Remarks:

- Unlike the first order linear ODE where we have an explicit expression for the solution, there is no explicit expression for the solution of second order linear ODE.
- Two integrations must be done to find solutions to second order linear. Therefore, initial value problems with two initial conditions can have a unique solution.

Existence and uniqueness of solutions.

Example

Find the longest interval $I \in \mathbb{R}$ such that there exists a unique solution to the initial value problem

$$(t-1)y'' - 3ty' + 4y = t(t-1), \qquad y(-2) = 2, \qquad y'(-2) = 1.$$

Solution: We first write the equation above in the form given in the Theorem above,

$$y'' - \frac{3t}{t-1}\, y' + \frac{4}{t-1}\, y = t.$$

The intervals where the hypotheses in the Theorem above are satisfied, that is, where the equation coefficients are continuous, are $I_1 = (-\infty, 1)$ and $I_2 = (1, \infty)$. Since the initial condition belongs to I_1, the solution domain is

$$I_1 = (-\infty, 1). \qquad \triangleleft$$

Existence and uniqueness of solutions.

Remarks:

- Every solution of the first order linear equation

$$y' + a(t)\,y = 0$$

is given by $y(t) = c\,e^{-A(t)}$, with $A(t) = \displaystyle\int a(t)\,dt$.

- All solutions above are proportional to each other:

$$y_1(t) = c_1\,e^{-A(t)}, \quad y_2(t) = c_2\,e^{-A(t)} \;\Rightarrow\; y_1(t) = \frac{c_1}{c_2}\,y_2(t)$$

Remark: The above statement is *not true* for solutions of second order, linear, homogeneous equations, $y'' + a_1(t)\,y' + a_0(t)y = 0$. Before we prove this statement we need few definitions:

- Proportional functions (linearly dependent).
- Wronskian of two functions.

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Linearly dependent and independent functions.

Definition

Two continuous functions y_1, $y_2 : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ are called *linearly dependent, (ld),* on the interval (t_1, t_2) iff there exists a constant c such that for all $t \in I$ holds

$$y_1(t) = c\, y_2(t).$$

The two functions are called *linearly independent, (li),* on the interval (t_1, t_2) iff they are not linearly dependent.

Remarks:

- y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ are ld $\Leftrightarrow$ there exist constants c_1, c_2, not both zero, such that $c_1\, y_1(t) + c_2\, y_2(t) = 0$ for all $t \in (t_1, t_2)$.
- y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ are li $\Leftrightarrow$ the only constants c_1, c_2, solutions of $c_1\, y_1(t) + c_2\, y_2(t) = 0$ for all $t \in (t_1, t_2)$ are $c_1 = c_2 = 0$.

Linearly dependent and independent functions.

Example

(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are ld.

(b) Show that $y_1(t) = \sin(t)$, $y_2(t) = t\sin(t)$ are li.

Solution:

Case (a): Trivial. $y_2 = 2y_1$.

Case (b): Find constants c_1, c_2 such that for all $t \in \mathbb{R}$ holds

$$c_1 \sin(t) + c_2 t \sin(t) = 0 \quad \Leftrightarrow \quad (c_1 + c_2 t) \sin(t) = 0.$$

Evaluating at $t = \pi/2$ and $t = 3\pi/2$ we obtain

$$c_1 + \frac{\pi}{2}\, c_2 = 0, \quad c_1 + \frac{3\pi}{2}\, c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.$$

We conclude: The functions y_1 and y_2 are li. $\triangleleft$

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The Wronskian of two functions.

Remark: The Wronskian is a function that determines whether two functions are ld or li.

Definition
The *Wronskian* of functions $y_1,\ y_2 : (t_1, t_2) \to \mathbb{R}$ is the function

$$W_{y_1 y_2}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Remark:

- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, then $W_{y_1 y_2}(t) = \det\big(A(t)\big)$.
- An alternative notation is: $W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

The Wronskian of two functions.

Example

Find the Wronskian of the functions:

(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$. (ld)

(b) $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t)$. (li)

Solution:

Case (a): $W_{y_1y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix}$. Therefore,

$$W_{y_1y_2}(t) = \sin(t)2\cos(t) - \cos(t)2\sin(t) \quad \Rightarrow \quad W_{y_1y_2}(t) = 0.$$

Case (b): $W_{y_1y_2} = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix}$. Therefore,

$$W_{y_1y_2}(t) = \sin(t)\big[\sin(t) + t\cos(t)\big] - \cos(t)t\sin(t).$$

We obtain $W_{y_1y_2}(t) = \sin^2(t)$. $\triangleleft$

The Wronskian of two functions.

Remark: The Wronskian determines whether two functions are linearly dependent or independent.

Theorem (Wronskian and linearly dependence)

The continuously differentiable functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ are linearly dependent iff $W_{y_1y_2}(t) = 0$ for all $t \in (t_1, t_2)$.

Remark: Importance of the Wronskian:

- Sometimes it is not simple to decide whether two functions are proportional to each other.
- The Wronskian is useful to study properties of solutions to ODE without having the explicit expressions of these solutions. (See Abel's Theorem later on.)

The Wronskian of two functions.

Example

Show whether the following two functions form a l.d. or l.i. set:

$$y_1(t) = \cos(2t) - 2\cos^2(t), \qquad y_2(t) = \cos(2t) + 2\sin^2(t).$$

Solution: Compute their Wronskian:

$$W_{y_1 y_2}(t) = y_1\, y_2' - y_1'\, y_2.$$

$$\begin{aligned} W_{y_1 y_2}(t) = &\big[\cos(2t) - 2\cos^2(t)\big]\,\big[-2\sin(2t) + 4\sin(t)\cos(t)\big] \\ &- \big[-2\sin(2t) + 4\sin(t)\cos(t)\big]\,\big[\cos(2t) + 2\sin^2(t)\big]. \end{aligned}$$

$$\sin(2t) = 2\sin(t)\cos(t) \Rightarrow \big[-2\sin(2t) + 4\sin(t)\cos(t)\big] = 0.$$

We conclude $W_{y_1 y_2}(t) = 0$, so the functions y_1 and y_2 are ld. ◁

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General and fundamental solutions.

Theorem

If a_1, $a_0 : (t_1, t_2) \to \mathbb{R}$ *are continuous, then the functions* $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ *solutions of the initial value problems*

$$\begin{aligned} &y_1'' + a_1(t)\, y_1' + a_0(t)\, y_1 = 0, \qquad && y_1(0) = 1, \quad && y_1'(0) = 0,\\ &y_2'' + a_1(t)\, y_2' + a_0(t)\, y_2 = 0, \qquad && y_2(0) = 0, \quad && y_2'(0) = 1, \end{aligned}$$

are linearly independent.

Remarks:

- Every linear combination $y(t) = c_1\, y_1(t) + c_2\, y_2(t)$, is also a solution of the differential equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0,$$

- Conversely, every solution y of the equation above can be written as a linear combination of the solutions y_1, y_2.

General and fundamental solutions.

Remark: The results above justify the following definitions.

Definition

Two solutions y_1, y_2 of the homogeneous equation

$$y'' + a_1(t)y' + a_0(t)y = 0, \tag{3}$$

are called *fundamental solutions* iff the functions y_1, y_2 are linearly independent, that is, iff $W_{y_1 y_2} \neq 0$.

Definition

Given any two fundamental solutions y_1, y_2, and arbitrary constants c_1, c_2, the function

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

is called the *general solution* of Eq. (3).

General and fundamental solutions.

Example

Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2\, y'' + 3t\, y' - y = 0.$$

Solution: First show that y_1 is a solution:

$$y_1 = t^{1/2}, \quad y_1' = \frac{1}{2}\, t^{-1/2}, \quad y_1'' = -\frac{1}{4}\, t^{-3/2},$$

$$2t^2 \left(-\frac{1}{4}\, t^{-\frac{3}{2}}\right) + 3t \left(\frac{1}{2} t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}} = -\frac{1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0.$$

Now show that y_2 is a solution:

$$y_2 = t^{-1}, \quad y_2' = -t^{-2}, \quad y_2'' = 2t^{-3},$$

$$2t^2 \left(2t^{-3}\right) + 3t \left(-t^{-2}\right) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0.$$

General and fundamental solutions.

Example

Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2\, y'' + 3t\, y' - y = 0.$$

Solution: We show that y_1, y_2 are linearly independent.

$$W_{y_1 y_2}(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}\, t^{-1/2} & -t^{-2} \end{vmatrix}.$$

$$W_{y_1 y_2}(t) = -t^{1/2}\, t^{-2} - \frac{1}{2} t^{-1/2}\, t^{-1} = -t^{-3/2} - \frac{1}{2} t^{-3/2}$$

$$W_{y_1 y_2}(t) = -\frac{3}{3}\, t^{-3/2} \quad \Rightarrow \quad y_1,\ y_2 \text{ li.} \qquad \triangleleft$$

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Abel's theorem on the Wronskian.

Theorem (Abel)

If a_1, $a_0 : (t_1, t_2) \to \mathbb{R}$ *are continuous functions and* y_1, y_2 *are continuously differentiable solutions of the equation*

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0,$$

then the Wronskian $W_{y_1 y_2}$ *is a solution of the equation*

$$W'_{y_1 y_2}(t) + a_1(t)\, W_{y_1 y_2}(t) = 0.$$

Therefore, for any $t_0 \in (t_1, t_2)$, *the Wronskian* $W_{y_1 y_2}$ *is given by*

$$W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, e^{A(t)} \qquad A(t) = \int_{t_0}^{t} a_1(s)\, ds.$$

Remarks: If the the Wronskian of two solutions vanishes at the initial time, then it vanishes at all times.

Abel's theorem on the Wronskian.

Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: Write the equation as in Abel's Theorem,

$$y'' - \Big(\frac{2}{t} + 1\Big)\, y' + \Big(\frac{2}{t^2} + \frac{1}{t}\Big)\, y = 0.$$

Abel's Theorem says that the Wronskian satisfies the equation

$$W'_{y_1y_2}(t) - \Big(\frac{2}{t} + 1\Big)\, W_{y_1y_2}(t) = 0.$$

This is a first order, linear equation for $W_{y_1y_2}$. The integrating factor method implies

$$A(t) = -\int_{t_0}^{t} \Big(\frac{2}{s} + 1\Big)\, ds = -2\ln\Big(\frac{t}{t_0}\Big) - (t - t_0)$$

Abel's theorem on the Wronskian.

Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: $A(t) = -2\ln\Big(\frac{t}{t_0}\Big) - (t - t_0) = \ln\Big(\frac{t_0^2}{t^2}\Big) - (t - t_0).$

The integrating factor is $\mu = \frac{t_0^2}{t^2}\, e^{-(t-t_0)}$. Therefore,

$$\Big[\mu(t) W_{y_1y_2}(t)\Big]' = 0 \quad \Rightarrow \quad \mu(t) W_{y_1y_2}(t) - \mu(t_0) W_{y_1y_2}(t_0) = 0$$

so, the solution is $W_{y_1y_2}(t) = W_{y_1y_2}(t_0)\, \frac{t^2}{t_0^2}\, e^{(t-t_0)}$.

Denoting $c = \big(W_{y_1y_2}(t_0)/t_0^2\big)\, e^{-t_0}$, then $W_{y_1y_2}(t) = c\, t^2 e^t$. $\triangleleft$

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Special Second order nonlinear equations

Definition

Given a functions $f : \mathbb{R}^3 \to \mathbb{R}$, *a second order* differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ is given by

$$y'' = f(t, y, y').$$

The equation is *linear* iff f is linear in the arguments y and y'.

Remarks:

- Nonlinear second order differential equation are usually difficult to solve.
- However, there are two particular cases where *second order* equations can be transformed into *first order* equations.
  - (a) $y'' = f(t, y')$. The function y is missing.
  - (b) $y'' = f(y, y')$. The independent variable t is missing.

Special Second order nonlinear equations

Remark: If second order differential equation has the form $y'' = f(t, y')$, then the equation for $v = y'$ is the first order equation $v' = f(t, v)$.

Example

Find the y solution of the second order nonlinear equation $y'' = -2t\,(y')^2$ with initial conditions $y(0) = 2$, $y'(0) = 1$.

Solution: Introduce $v = y'$. Then $v' = y''$, and

$$v' = -2t\,v^2 \quad \Rightarrow \quad \frac{v'}{v^2} = -2t \quad \Rightarrow \quad -\frac{1}{v} = -t^2 + c.$$

So, $\dfrac{1}{y'} = t^2 - c$, that is, $y' = \dfrac{1}{t^2 - c}$. The initial condition implies

$$1 = y'(0) = -\frac{1}{c} \quad \Rightarrow \quad c = -1 \quad \Rightarrow \quad y' = \frac{1}{t^2 - 1}.$$

Special Second order nonlinear equations

Example

Find the y solution of the second order nonlinear equation $y'' = -2t\,(y')^2$ with initial conditions $y(0) = 2$, $y'(0) = 1$.

Solution: Then, $y = \displaystyle\int \frac{dt}{t^2 - 1} + c$. Partial Fractions!

$$\frac{1}{t^2 - 1} = \frac{1}{(t-1)(t+1)} = \frac{a}{(t-1)} + \frac{b}{(t+1)}.$$

Hence, $1 = a(t+1) + b(t-1)$. Evaluating at $t = 1$ and $t = -1$ we get $a = \dfrac{1}{2}$, $b = -\dfrac{1}{2}$. So $\dfrac{1}{t^2 - 1} = \dfrac{1}{2}\Big[\dfrac{1}{(t-1)} - \dfrac{1}{(t+1)}\Big]$.

$$y = \frac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + c. \qquad 2 = y(0) = \frac{1}{2}(0 - 0) + c.$$

We conclude $y = \dfrac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + 2$. ◁

Special Second order nonlinear equations

Remark: We now consider the case (b) $y'' = f(y, y')$. The independent variable t is missing.

Theorem

Consider a second order differential equation $y'' = f(y, y')$, *and introduce the function* $v(t) = y'(t)$. *If the function* y *is invertible, then the new function* $\hat{v}(y) = v(t(y))$ *satisfies the first order differential equation*

$$\frac{d\hat{v}}{dy} = \frac{1}{\hat{v}}\, f(y, \hat{v}(y)).$$

Proof: Notice that $v'(t) = f(y, v(t))$. Now, by chain rule

$$\frac{d\hat{v}}{dy}\Big|_{y} = \frac{dv}{dt}\Big|_{t(y)} \frac{dt}{dy}\Big|_{t(y)} = \frac{v'}{y'}\Big|_{t(y)} = \frac{v'}{v}\Big|_{t(y)} = \frac{f\big(y, v\big)}{v}\Big|_{t(y)}.$$

Therefore, $\dfrac{d\hat{v}}{dy} = \dfrac{1}{\hat{v}} f(y, \hat{v}(y))$. □

Special Second order nonlinear equations

Example

Find a solution y to the second order equation $y'' = 2y\, y'$.

Solution: The variable t does not appear in the equation.
Hence, $v(t) = y'(t)$. The equation is $v'(t) = 2y(t)\, v(t)$.
Now introduce $\hat{v}(y) = v(t(y))$. Then

$$\frac{d\hat{v}}{dy} = \Big(\frac{dv}{dt}\,\frac{dt}{dy}\Big)\Big|_{t(y)} = \frac{v'}{y'}\Big|_{t(y)} = \frac{v'}{v}\Big|_{t(y)}.$$

Using the differential equation,

$$\frac{d\hat{v}}{dy} = \frac{2yv}{v}\Big|_{t(y)} \quad \Rightarrow \quad \frac{d\hat{v}}{dy} = 2y \quad \Rightarrow \quad \hat{v}(y) = y^2 + c.$$

Since $v(t) = \hat{v}(y(t))$, we get $v(t) = y^2(t) + c$.

Special Second order nonlinear equations

Example

Find a solution y to the second order equation $y'' = 2y\,y'$.

Solution: Recall: $v(t) = y^2(t) + c$. This is a separable equation,

$$\frac{y'(t)}{y^2(t) + c} = 1.$$

Since we only need to find a solution of the equation, and the integral depends on whether $c > 0$, $c = 0$, $c < 0$, we choose (for no special reason) only one case, $c = 1$.

$$\int \frac{dy}{1 + y^2} = \int dt + c_0 \quad \Rightarrow \quad \arctan(y) = t + c_0 y(t) = \tan(t + c_0).$$

Again, for no reason, we choose $c_0 = 0$, and we conclude that one possible solution to our problem is $y(t) = \tan(t)$. $\triangleleft$