Complex, distinct eigenvalues (Sect. 5.8)

- Review: The case of diagonalizable matrices.
- Classification of 2×2 systems.
- Real matrix with a pair of complex eigenvalues.
- Phase portraits for 2×2 systems.

Review: The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

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Theorem

If an $n \times n$ matrix A has n distinct eigenvalues, then matrix A is diagonalizable.

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Review: Classification of 2×2 systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

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(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

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(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

Review: Classification of 2×2 systems.

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(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

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(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9).

Complex, distinct eigenvalues (Sect. 5.8)

- Review: The case of diagonalizable matrices.
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Real matrix with a pair of complex eigenvalues.

Theorem

If $\{\lambda, \mathbf{v}\}$ *is an eigen-pair of an* $n \times n$ *real-valued matrix* A*, then* $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ *also is an eigen-pair of matrix* A*.*

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Therefore $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ is an eigen-pair of matrix A. $\square$

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Remark: The Theorem above is equivalent to the following:

Real matrix with a pair of complex eigenvalues.

Theorem

If $\{\lambda, \mathbf{v}\}$ is an eigen-pair of an $n \times n$ real-valued matrix A, then $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ also is an eigen-pair of matrix A.

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$$\overline{A\,\mathbf{v}} = \overline{\lambda\,\mathbf{v}} \quad \Leftrightarrow \quad \overline{A}\,\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}} \quad \Leftrightarrow \quad A\,\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}.$$

Therefore $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ is an eigen-pair of matrix A. □

Remark: The Theorem above is equivalent to the following:
If an $n \times n$ real-valued matrix A has eigen pairs

$$\lambda_1 = \alpha + i\beta, \quad \mathbf{v}_1 = \mathbf{a} + i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then so is

$$\lambda_2 = \alpha - i\beta, \quad \mathbf{v}_2 = \mathbf{a} - i\mathbf{b}.$$

Real matrix with a pair of complex eigenvalues.

Theorem (Complex pairs)

If an $n \times n$ real-valued matrix A has eigen pairs

$$\lambda_{\pm} = \alpha \pm i\beta, \quad \mathbf{v}^{(\pm)} = \mathbf{a} \pm i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then the differential equation

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

has a linearly independent set of two complex-valued solutions

$$\mathbf{x}^{(+)} = \mathbf{v}^{(+)}\, e^{\lambda_+ t}, \qquad \mathbf{x}^{(-)} = \mathbf{v}^{(-)}\, e^{\lambda_- t},$$

and it also has a linearly independent set of two real-valued solutions

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\, \cos(\beta t) - \mathbf{b}\, \sin(\beta t)\big]\, e^{\alpha t},$$

$$\mathbf{x}^{(2)} = \big[\mathbf{a}\, \sin(\beta t) + \mathbf{b}\, \cos(\beta t)\big]\, e^{\alpha t}.$$

Real matrix with a pair of complex eigenvalues.

Proof: We know that one solution to the differential equation is

$$\mathbf{x}^{(+)} = \mathbf{v}^{(+)} \, e^{\lambda_+ t}$$

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Euler equation implies

Real matrix with a pair of complex eigenvalues.

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Euler equation implies

$$\mathbf{x}^{(+)} = (\mathbf{a} + i\mathbf{b})\, e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t)\right],$$

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A similar calculation done on $\mathbf{x}^{(-)}$ implies

$$\mathbf{x}^{(-)} = \left[\mathbf{a}\, \cos(\beta t) - \mathbf{b}\, \sin(\beta t)\right] e^{\alpha t} - i \left[\mathbf{a}\, \sin(\beta t) + \mathbf{b}\, \cos(\beta t)\right] e^{\alpha t}.$$

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Introduce $\mathbf{x}^{(1)} = (\mathbf{x}^{(+)} + \mathbf{x}^{(-)})/2$,

Real matrix with a pair of complex eigenvalues.

Proof: We know that one solution to the differential equation is

$$\mathbf{x}^{(+)} = \mathbf{v}^{(+)}\, e^{\lambda_+ t} = (\mathbf{a} + i\mathbf{b})\, e^{(\alpha+i\beta)t} = (\mathbf{a} + i\mathbf{b})\, e^{\alpha t}\, e^{i\beta t}.$$

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Introduce $\mathbf{x}^{(1)} = (\mathbf{x}^{(+)} + \mathbf{x}^{(-)})/2$, $\mathbf{x}^{(2)} = (\mathbf{x}^{(+)} - \mathbf{x}^{(-)})/(2i)$,

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$$\mathbf{x}^{(+)} = (\mathbf{a} + i\mathbf{b})\, e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t)\right],$$

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Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Real matrix with a pair of complex eigenvalues.

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The roots of the characteristic polynomial are

$(\lambda - 2)^2 + 9 = 0$

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$$(\lambda - 2)^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} - 2 = \pm 3i$$

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(2) Find the eigenvectors of matrix A above. For λ_+,

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$$A - \lambda_+ I = A - (2+3i)I = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}.$$

Real matrix with a pair of complex eigenvalues.

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We need to solve $(A - \lambda_+ I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$.

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We need to solve $(A - \lambda_+ I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$. Gauss operations

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$$

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Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: $\lambda_{\pm} = 2 \pm 3i$, $(A - \lambda_{+} I) = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}$.

We need to solve $(A - \lambda_{+} I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$. Gauss operations

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: $\lambda_{\pm} = 2 \pm 3i$, $(A - \lambda_+ I) = \begin{bmatrix} 2 - (2+3i) & 3 \\ -3 & 2 - (2+3i) \end{bmatrix}$.

We need to solve $(A - \lambda_+ I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$. Gauss operations

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix}$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: $\lambda_{\pm} = 2 \pm 3i$, $(A - \lambda_+ I) = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}$.

We need to solve $(A - \lambda_+ I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$. Gauss operations

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$

So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: $\lambda_\pm = 2 \pm 3i$, $(A - \lambda_+ I) = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}$.

We need to solve $(A - \lambda_+ I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$. Gauss operations

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$

So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$.

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: $\lambda_{\pm} = 2 \pm 3i$, $(A - \lambda_+ I) = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}$.

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So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$. Choose

$$v_2 = 1, \quad v_1 = -i,$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$

So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$. Choose

$$v_2 = 1, \quad v_1 = -i, \quad \Rightarrow \quad \mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_+ = 2 + 3i.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$,

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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Real matrix with a pair of complex eigenvalues.

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The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$, that is, $\mathbf{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Notice that $\mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

Real matrix with a pair of complex eigenvalues.

Example

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Notice that $\mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$, that is, $\mathbf{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Notice that $\mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$ implies

$$\alpha = 2,$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$, that is, $\mathbf{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Notice that $\mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$ implies

$$\alpha = 2, \qquad \beta = 3,$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$, that is, $\mathbf{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

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$$\alpha = 2, \qquad \beta = 3, \qquad \mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Real matrix with a pair of complex eigenvalues.

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$$\alpha = 2, \qquad \beta = 3, \qquad \mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Real matrix with a pair of complex eigenvalues.

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Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}$, and

Real matrix with a pair of complex eigenvalues.

Example

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$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\,e^{\alpha t}$, and $\mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\,e^{\alpha t}$.

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}$, and $\mathbf{x}^{(2)} = \big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big]\, e^{\alpha t}$. That is

$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t}$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}$, and $\mathbf{x}^{(2)} = \big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big]\, e^{\alpha t}$. That is

$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}$, and $\mathbf{x}^{(2)} = \big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big]\, e^{\alpha t}$. That is

$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}.$$

$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \right) e^{2t}$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t)\big]\, e^{\alpha t}$, and $\mathbf{x}^{(2)} = \big[\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t)\big]\, e^{\alpha t}$. That is

$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}.$$

$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \right) e^{2t} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}.$$

$\triangleleft$

Complex, distinct eigenvalues (Sect. 5.8)

- Review: The case of diagonalizable matrices.
- Classification of 2×2 systems.
- Real matrix with a pair of complex eigenvalues.
- **Phase portraits for 2×2 systems.**

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

The phase portrait of the vectors

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

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$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

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Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

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$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

$$\tilde{\mathbf{x}}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix},$$

is a radius one circle.

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

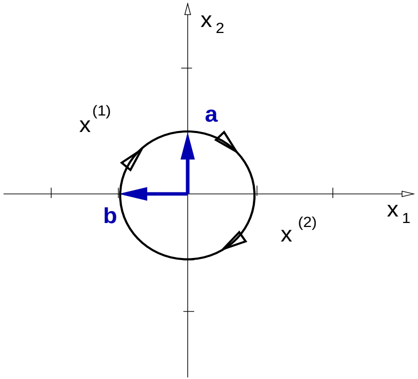
Solution:

The phase portrait of the vectors

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

$$\tilde{\mathbf{x}}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix},$$

is a radius one circle.



Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

The phase portrait of the solutions

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

$$\tilde{\mathbf{x}}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t},$$

are outgoing spirals.

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

The phase portrait of the solutions

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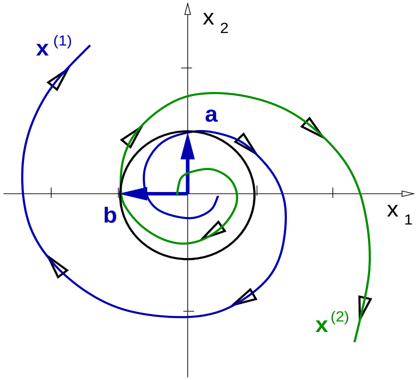


Phase portraits for 2×2 systems.

Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t}, \quad \mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Phase portraits for 2×2 systems.

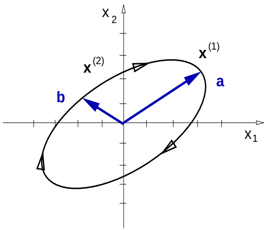
Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \left[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\right] e^{\alpha t}, \ \mathbf{x}^{(2)} = \left[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\right] e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Solution:



Phase portraits for 2 × 2 systems.

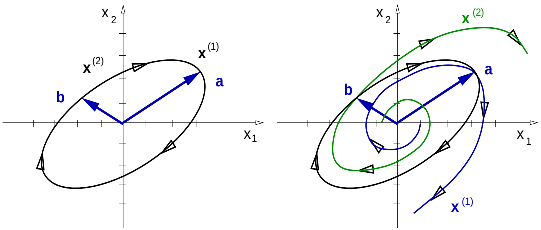
Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}, \quad \mathbf{x}^{(2)} = \big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big]\, e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Solution:



Phase portraits for 2×2 systems.

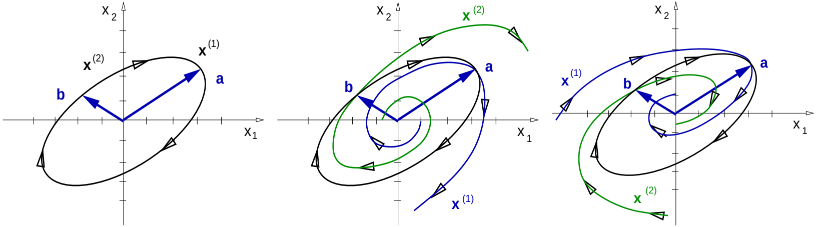
Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t},\ \mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Solution:



◁

Real, repeated eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Repeated eigenvalue diagonalizable 2×2 system.
- Repeated eigenvalue non-diagonalizable 2×2 system.
- Phase portraits for 2×2 systems.

Review: Classification of 2×2 systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

Review: Classification of 2×2 systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

Review: Classification of 2×2 systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

Review: Classification of 2×2 systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Review: Classification of 2×2 systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9).

Real, repeated eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- **Repeated eigenvalue diagonalizable 2×2 system.**
- Repeated eigenvalue non-diagonalizable 2×2 system.
- Phase portraits for 2×2 systems.

Repeated eigenvalue diagonalizable 2×2 system.

Remark: For 2×2 systems the situation is fairly simple.

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Every 2×2 diagonalizable matrix A with repeated eigenvalue λ has the form $A = \lambda I$.

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Proof: Since A is diagonalizable, exists P invertible such that

$$A = P \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} P^{-1}$$

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$$\mathbf{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}$$

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Remark: The solution phase portraits are always straight lines passing through the origin.

Real, repeated eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Repeated eigenvalue diagonalizable 2×2 system.
- **Repeated eigenvalue non-diagonalizable 2×2 system.**
- Phase portraits for 2×2 systems.

Repeated eigenvalue non-diagonalizable 2×2 system.

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(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Repeated eigenvalue non-diagonalizable 2×2 system.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

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Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9). Next Class.

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Show that matrix $B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Show that matrix $B = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Solution: We need to show that all eigenvectors of matrix B are proportional to each other.

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Show that matrix $B = \dfrac{1}{2} \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Solution: We need to show that all eigenvectors of matrix B are proportional to each other. We start computing the eigenvalues,

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Show that matrix $B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Solution: We need to show that all eigenvectors of matrix B are proportional to each other. We start computing the eigenvalues,

$$p(\lambda) = \det(B - \lambda I)$$

Repeated eigenvalue non-diagonalizable 2×2 system.

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$$p(\lambda) = \det(B - \lambda I) = \begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{vmatrix}$$

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$$p(\lambda) = \lambda^2 - 4\lambda + 4 = 0$$

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$$p(\lambda) = \lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = 2.$$

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We now compute the corresponding eigenvectors,

Repeated eigenvalue non-diagonalizable 2×2 system.

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Repeated eigenvalue non-diagonalizable 2×2 system.

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We now compute the corresponding eigenvectors,

$$(B - 2I) = \; = \begin{bmatrix} \frac{3}{2} - 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - 2 \end{bmatrix}$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Show that matrix $B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Solution: We need to show that all eigenvectors of matrix B are proportional to each other. We start computing the eigenvalues,

$$p(\lambda) = \det(B - \lambda I) = \begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{vmatrix} = \Big(\frac{3}{2} - \lambda\Big)\Big(\frac{5}{2} - \lambda\Big) + \frac{1}{4}$$

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Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Show that matrix $B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

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We now compute the corresponding eigenvectors,

$$(B - 2I) = \; = \begin{bmatrix} \frac{3}{2} - 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \rightarrow$$

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Example

Show that matrix $B = \dfrac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

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We now compute the corresponding eigenvectors,

$$(B - 2I) = \; = \begin{bmatrix} \frac{3}{2} - 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Show that matrix $B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Solution: We need to show that all eigenvectors of matrix B are proportional to each other. We start computing the eigenvalues,

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We now compute the corresponding eigenvectors,

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Hence all eigenvectors are proportional to $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\triangleleft$

Repeated eigenvalue non-diagonalizable 2×2 system.

Theorem (Repeated eigenvalue)

If λ is an eigenvalue of an $n \times n$ matrix A having algebraic multiplicity $r = 2$ and only one associated eigen-direction, then the differential equation

$$\mathbf{x}'(t) = A\,\mathbf{x}(t),$$

has a linearly independent set of solutions given by

$$\left\{\mathbf{x}^{(1)}(t) = \mathbf{v}\; e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \left(\mathbf{v}\; t + \mathbf{w}\right) e^{\lambda t}\right\}.$$

where the vector $\mathbf{w}$ is solution of

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

which always has a solution $\mathbf{w}$.

Repeated eigenvalue non-diagonalizable 2×2 system.

Recall: The case of a single second order equation

$$y'' + a_1\, y' + a_0\, y = 0$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Recall: The case of a single second order equation

$$y'' + a_1\, y' + a_0\, y = 0$$

with characteristic polynomial

$$p(r) = r^2 + a_1\, r + a_0$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Recall: The case of a single second order equation

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$$p(r) = r^2 + a_1\, r + a_0 = (r - r_1)^2.$$

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$$p(r) = r^2 + a_1\, r + a_0 = (r - r_1)^2.$$

In this case a fundamental set of solutions is

$$\big\{ y_1(t) = e^{r_1 t}, \quad y_2(t) = t\, e^{r_1 t} \big\}.$$

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Recall: The case of a single second order equation

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This is not the case with systems of first order linear equations,

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$$\big\{ \mathbf{x}^{(1)}(t) = \mathbf{v}\ e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\ t + \mathbf{w}\big)\, e^{\lambda t} \big\}.$$

In general, $\mathbf{w} \neq \mathbf{0}$.

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A.

Repeated eigenvalue non-diagonalizable 2×2 system.

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Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A. Its characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(-\frac{3}{2} - \lambda\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} - \lambda\right) \end{vmatrix}$$

Repeated eigenvalue non-diagonalizable 2×2 system.

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So $p(\lambda) = \lambda^2 + 2\lambda + 1$

Repeated eigenvalue non-diagonalizable 2×2 system.

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So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$.

Repeated eigenvalue non-diagonalizable 2×2 system.

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So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. The roots and multiplicity are

$$\lambda = -1, \qquad r = 2.$$

Repeated eigenvalue non-diagonalizable 2×2 system.

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The corresponding eigenvectors are the solutions of $(A + I)\mathbf{v} = \mathbf{0}$,

Repeated eigenvalue non-diagonalizable 2×2 system.

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So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. The roots and multiplicity are

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The corresponding eigenvectors are the solutions of $(A + I)\mathbf{v} = \mathbf{0}$,

$$\begin{bmatrix} \left(-\frac{3}{2} + 1\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} + 1\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A. Its characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(-\frac{3}{2} - \lambda\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} - \lambda\right) \end{vmatrix} = \left(\lambda + \frac{3}{2}\right)\left(\lambda + \frac{1}{2}\right) + \frac{1}{4}.$$

So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. The roots and multiplicity are

$$\lambda = -1, \qquad r = 2.$$

The corresponding eigenvectors are the solutions of $(A + I)\mathbf{v} = \mathbf{0}$,

$$\begin{bmatrix} \left(-\frac{3}{2} + 1\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} + 1\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

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Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Recall: $\lambda = -1$, with $r = 2$, and $(A + I) \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

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The eigenvector components satisfy: $v_1 = 2v_2$.

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$$\lambda = -1, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2.$$

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We conclude that this eigenvalue has only one eigen-direction.

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Theorem above says we need to find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

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$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right]$$

Repeated eigenvalue non-diagonalizable 2×2 system.

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$\lambda = -1$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2$, and $(A + I)\mathbf{w} = \mathbf{v} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right]$.

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We obtain $w_1 = 2w_2 - 4$.

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Repeated eigenvalue non-diagonalizable 2×2 system.

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Given a solution $\mathbf{w}$, then $c\mathbf{v} + \mathbf{w}$ is also a solution, $c \in \mathbb{R}$.

Repeated eigenvalue non-diagonalizable 2×2 system.

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We choose the simplest solution, $\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

Repeated eigenvalue non-diagonalizable 2×2 system.

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Given a solution $\mathbf{w}$, then $c\mathbf{v} + \mathbf{w}$ is also a solution, $c \in \mathbb{R}$.

We choose the simplest solution, $\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$. We conclude,

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}. \qquad \triangleleft$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4} \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Repeated eigenvalue non-diagonalizable 2×2 system.

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Find the solution $\mathbf{x}$ to the IVP

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Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

Repeated eigenvalue non-diagonalizable 2×2 system.

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Find the solution $\mathbf{x}$ to the IVP

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The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Repeated eigenvalue non-diagonalizable 2×2 system.

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Find the solution $\mathbf{x}$ to the IVP

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The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$

Repeated eigenvalue non-diagonalizable 2×2 system.

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Find the solution $\mathbf{x}$ to the IVP

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The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4} \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

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Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find the solution $\mathbf{x}$ to the IVP

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We conclude: $\mathbf{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{4}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$ $\lhd$

Real, repeated eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Repeated eigenvalue diagonalizable 2×2 system.
- Repeated eigenvalue non-diagonalizable 2×2 system.
- **Phase portraits for 2×2 systems.**

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of

$$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Phase portraits for 2×2 systems.

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$$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution:

We start plotting the vectors

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

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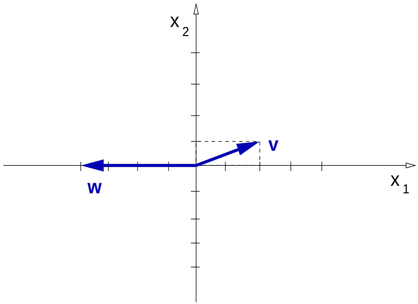

Phase portraits for 2×2 systems.

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$$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

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Now plot the solutions

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$$

$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

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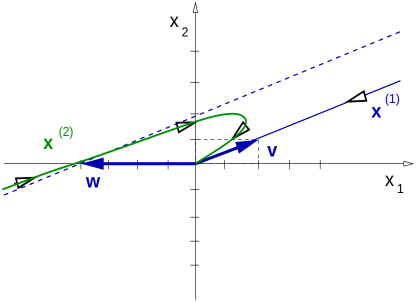

Phase portraits for 2×2 systems.

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Now plot the solutions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

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Phase portraits for 2×2 systems.

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This is the case $\lambda < 0$.

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of

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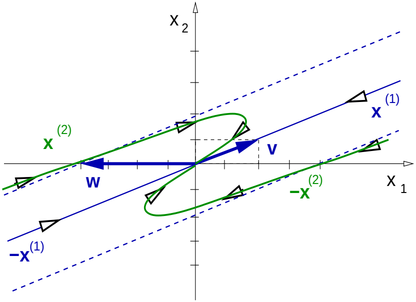

Phase portraits for 2×2 systems.

Example

Given any vectors $\mathbf{v}$ and $\mathbf{w}$, and any constant λ, plot the phase portraits of the functions

$$\mathbf{x}^{(1)}(t) = \mathbf{v}\, e^{\lambda t}, \qquad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{\lambda t},$$

Solution:

The case $\lambda < 0$. We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

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Phase portraits for 2×2 systems.

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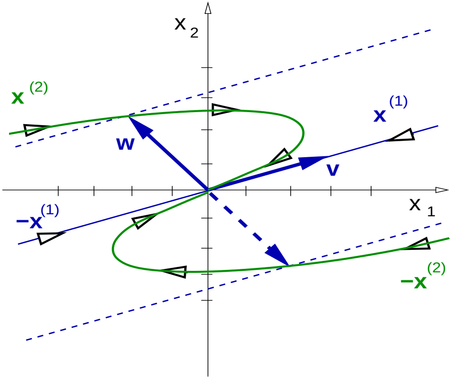

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Solution:

The case $\lambda > 0$. We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

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Phase portraits for 2×2 systems.

Example

Given any vectors $\mathbf{v}$ and $\mathbf{w}$, and any constant λ, plot the phase portraits of the functions

$$\mathbf{x}^{(1)}(t) = \mathbf{v}\, e^{\lambda t}, \qquad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{\lambda t},$$

Solution:

The case $\lambda > 0$. We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$