Solving the Heat Equation (Sect. 6.3).

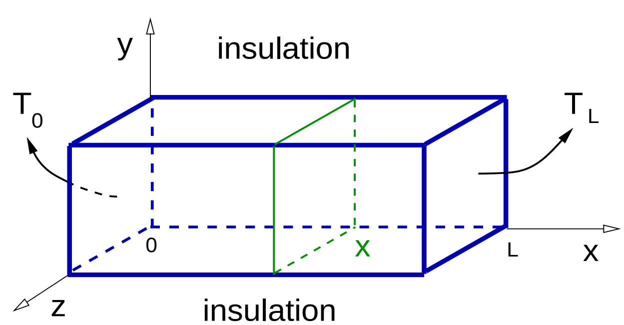
- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.

The Heat Equation.

Review: The Heat Equation describes the temperature distribution in a solid material as function of time and position in space.

Problem: Find the temperature, u, of a bar of length L with insulated horizontal sides and vertical extremes kept at fixed temperatures u_0, u_L, and with initial temperature $u(0,x) = f(x)$.

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x), \quad x \in (0,L),$$
$$u(0) = T_0, \quad u(L) = T_L, \quad u(0,x) = f(x).$$



Remark: The heat transfer occurs only along the x-axis.

The Heat Equation.

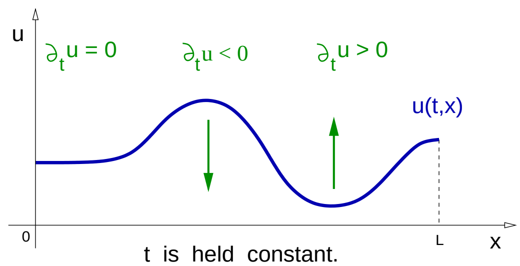
Remarks:

- The unknown of the problem is $u(t,x)$, the temperature of the bar at the time t and position x.
- The temperature does not depend on y or z.
- The one-dimensional Heat Equation is:

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x),$$

where $k > 0$ is the heat conductivity, units: $[k] = \dfrac{(\text{distance})^2}{(\text{time})}$.

- The Heat Equation is a Partial Differential Equation, PDE.



Solving the Heat Equation (Sect. 6.3).

- The Heat Equation.
- **The Initial-Boundary Value Problem.**
- The separation of variables method.
- An example of separation of variables.

The Initial-Boundary Value Problem.

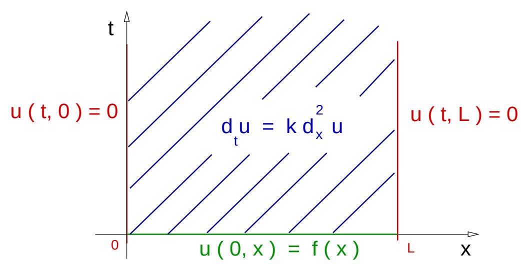
Definition

The IBVP for the one-dimensional Heat Equation is the following: Given a constant $k > 0$ and a function $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \to \mathbb{R}$ solution of

$$\partial_t u(t,x) = k\, \partial_x^2 u(t,x),$$

$$\text{I.C.:} \quad u(0,x) = f(x),$$

$$\text{B.C.:} \quad u(t,0) = 0, \quad u(t,L) = 0.$$



Solving the Heat Equation (Sect. 6.3).

- The Heat Equation.
- The Initial-Boundary Value Problem.
- **The separation of variables method.**
- An example of separation of variables.

The separation of variables method.

Summary: IBVP for the Heat Equation.

The vector Space: (Functions of x.)
$V = \{v$ differentiable functions on $[0, L]$, with $v(0) = v(L) = 0\}$.

Remark: The problem B.C. are imbedded in the definition of V.

The orthogonal vector basis: (Functions of x.)
Introduce $\{w_n\}_{n=1}^{\infty} \subset V$, that is, $w_n(0) = w_n(L) = 0$.

Remark: The basis is not known yet. Finding the basis is part of our problem.

Decompose the temperature u in the basis $\{w_n\}$:

$$u(t, x) = \sum_{n=1}^{\infty} v_n(t)\, w_n(x).$$

We need to find all v_n and w_n.

The separation of variables method.

Recall: $u(t, x) = \sum_{n=1}^{\infty} v_n(t)\, w_n(x)$.

Introduce u into the differential equation.

$$\sum_{n=1}^{\infty} \big[\partial_t(v_n w_n) - k\, \partial_x^2(v_n w_n)\big] = 0.$$

A sufficient condition to find a solution is: Each term vanishes.

$$\partial_t(v_n w_n) = k\, \partial_x^2(v_n w_n).$$

But v_n depends on t and w_n depends on x.
Denote $\partial_t v_n = \dot{v}_n$, and $\partial_x w_n = w_n'$. Then, for each $n \geqslant 1$,

$$\dot{v}_n(t)\, w_n(x) = k\, v_n(t)\, w_n''(x).$$

$$\frac{1}{k} \frac{\dot{v}_n(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)}.$$

The separation of variables method.

Recall: $\dfrac{1}{k}\dfrac{\dot v_n(t)}{v_n(t)} = \dfrac{w_n''(x)}{w_n(x)}$. But:

$$\frac{1}{k}\,\frac{\dot v_n(t)}{}\frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\,\frac{d^2 w_n}{dx^2}(x).$$

Depends only on t = Depends only on x.

- The left-hand side depends only on t, while the right-hand side depends only on x.
- When this happens in a PDE, one can use the separation of variables method on that PDE.
- The conclusion is: Each side must be constant; $-\lambda_n$.
$$\frac{1}{k}\,\frac{\dot v_n(t)}{v_n(t)} = -\lambda_n, \qquad \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$
- The PDE is transformed into infinitely many ODEs.

The separation of variables method.

Recall: $\dfrac{1}{k}\,\dfrac{\dot v_n(t)}{v_n(t)} = -\lambda_n, \qquad \dfrac{w_n''(x)}{w_n(x)} = -\lambda_n.$

The equation for v_n is linear,

$$\dot v_n = -k\lambda_n\, v_n \quad \Rightarrow \quad v_n(t) = v_n(0)\, e^{-k\lambda_n t}.$$

The equation for w_n is linear too, it is a BVP:

$$w_n'' + \lambda_n\, w_n = 0, \quad w_n(0) = w_n(L) = 0.$$

We have solved these eigenfunction problems before:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad w_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

We have seen that these functions form an orthogonal set.

The separation of variables method.

Conclusion: $u(t,x) = \sum_{n=1}^{\infty} v_n(0)\, e^{-k(\frac{n\pi}{L})t} \sin\Big(\frac{n\pi x}{L}\Big).$

This function satisfies the Boundary condtions:

$$u(t,0) = u(t,L) = 0.$$

It must satisfy the initial condition:

$$f(x) = u(0,x) = \sum_{n=1}^{\infty} v_n(0)\, \sin\Big(\frac{n\pi x}{L}\Big).$$

But w_n are orthogonal with: $\displaystyle\int_0^L \sin^2\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{L}{2}.$

$$\int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = v_n(0)\, \frac{L}{2}.$$

The separation of variables method.

Conclusion: $u(t,x) = \sum_{n=1}^{\infty} v_n(0)\, e^{-k(\frac{n\pi}{L})t} \sin\Big(\frac{n\pi x}{L}\Big).$

With the coefficients $v_n(0)$ given by:

$$v_n(0) = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

Summary:IBVP for the Heat Equation.

Decompose: $u(t,x) = \sum_{n=1}^{\infty} v_n(t)\, w_n(x)$, where:

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.
- $v_n(0)$: Fourier Series coefficients.

Remark: Separation of variables does not work for every PDE.

Solving the Heat Equation (Sect. 6.3).

- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- **An example of separation of variables.**

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

We solve for v_n with the initial condition $v_n(0) = 1$.

$$e^{\frac{\lambda_n}{4}t}\, v_n'(t) + \frac{\lambda_n}{4}\, e^{\frac{\lambda_n}{4}t}\, v_n(t) = 0 \quad \Rightarrow \quad \big[e^{\frac{\lambda_n}{4}t}\, v_n(t)\big]' = 0.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \qquad w_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

$$u(t, x) = \sum_{n=1}^{\infty} c_n\, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$

The initial condition is $3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right).$

The orthogonality of the sine functions implies

$$3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx.$$

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$. Therefore,

$$3\sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad c_1 = 3.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: We conclude that

$$u(t, x) = 3 \, e^{-(\frac{\pi}{4})^2 t} \sin\left(\frac{\pi x}{2}\right).$$