Complex, distinct eigenvalues (Sect. 5.8)

- Review: The case of diagonalizable matrices.
- Classification of 2 × 2 systems.
- Real matrix with a pair of complex eigenvalues.
- Phase portraits for 2 × 2 systems.

Review: The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Theorem

If an $n \times n$ matrix A has n distinct eigenvalues, then matrix A is diagonalizable.

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Review: Classification of 2×2 systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9).

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Real matrix with a pair of complex eigenvalues.

Theorem

If $\{\lambda, \mathbf{v}\}$ is an eigen-pair of an $n \times n$ real-valued matrix A, then $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ also is an eigen-pair of matrix A.

Proof: By hypothesis $A\,\mathbf{v} = \lambda\,\mathbf{v}$ and $\overline{A} = A$. Then

$$\overline{A\,\mathbf{v}} = \overline{\lambda\,\mathbf{v}} \quad \Leftrightarrow \quad \overline{A}\,\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}} \quad \Leftrightarrow \quad A\,\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}.$$

Therefore $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ is an eigen-pair of matrix A. $\square$

Remark: The Theorem above is equivalent to the following:
If an $n \times n$ real-valued matrix A has eigen pairs

$$\lambda_1 = \alpha + i\beta, \quad \mathbf{v}_1 = \mathbf{a} + i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then so is

$$\lambda_2 = \alpha - i\beta, \quad \mathbf{v}_2 = \mathbf{a} - i\mathbf{b}.$$

Real matrix with a pair of complex eigenvalues.

Theorem (Complex pairs)

If an $n \times n$ *real-valued matrix* A *has eigen pairs*

$$\lambda_{\pm} = \alpha \pm i\beta, \quad \mathbf{v}^{(\pm)} = \mathbf{a} \pm i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ *and* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, *then the differential equation*

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

has a linearly independent set of two complex-valued solutions

$$\mathbf{x}^{(+)} = \mathbf{v}^{(+)}\, e^{\lambda_+ t}, \qquad \mathbf{x}^{(-)} = \mathbf{v}^{(-)}\, e^{\lambda_- t},$$

and it also has a linearly independent set of two real-valued solutions

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t},$$

$$\mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}.$$

Real matrix with a pair of complex eigenvalues.

Proof: We know that one solution to the differential equation is

$$\mathbf{x}^{(+)} = \mathbf{v}^{(+)}\, e^{\lambda_+ t} = (\mathbf{a} + i\mathbf{b})\, e^{(\alpha + i\beta)t} = (\mathbf{a} + i\mathbf{b})\, e^{\alpha t}\, e^{i\beta t}.$$

Euler equation implies

$$\mathbf{x}^{(+)} = (\mathbf{a} + i\mathbf{b})\, e^{\alpha t}\, \big[\cos(\beta t) + i\sin(\beta t)\big],$$

$$\mathbf{x}^{(+)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t} + i\,\big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}$$

A similar calculation done on $\mathbf{x}^{(-)}$ implies

$$\mathbf{x}^{(-)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t} - i\,\big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}.$$

Introduce $\mathbf{x}^{(1)} = (\mathbf{x}^{(+)} + \mathbf{x}^{(-)})/2$, $\mathbf{x}^{(2)} = (\mathbf{x}^{(+)} - \mathbf{x}^{(-)})/(2i)$, then

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t},$$

$$\mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}. \qquad \square$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: (1) Find the eigenvalues of matrix A above,

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} (2-\lambda) & 3 \\ -3 & (2-\lambda) \end{vmatrix} = (\lambda - 2)^2 + 9.$$

The roots of the characteristic polynomial are

$$(\lambda - 2)^2 + 9 = 0 \quad \Rightarrow \quad \lambda_\pm - 2 = \pm 3i \quad \Rightarrow \quad \lambda_\pm = 2 \pm 3i.$$

(2) Find the eigenvectors of matrix A above. For λ_+,

$$A - \lambda_+ I = A - (2+3i)I = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: $\lambda_\pm = 2 \pm 3i$, $(A - \lambda_+ I) = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}.$

We need to solve $(A - \lambda_+ I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$. Gauss operations

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$

So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$. Choose

$$v_2 = 1, \quad v_1 = -i, \quad \Rightarrow \quad \mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_+ = 2 + 3i.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$, that is, $\mathbf{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Notice that $\mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$ implies

$$\alpha = 2, \qquad \beta = 3, \qquad \mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: $\alpha = 2$, $\beta = 3$, $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t)\big]\, e^{\alpha t}$, and $\mathbf{x}^{(2)} = \big[\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t)\big]\, e^{\alpha t}$. That is

$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}.$$

$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \right) e^{2t} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}.$$

$\lhd$

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Phase portraits for 2×2 systems.

Example

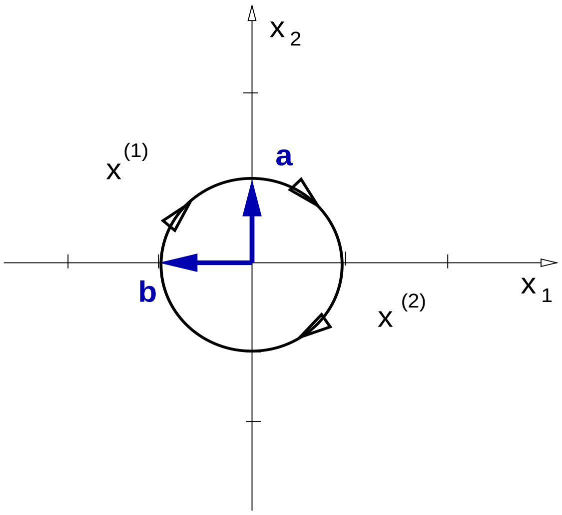
Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:
The phase portrait of the vectors

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

$$\tilde{\mathbf{x}}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix},$$

is a radius one circle.



Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

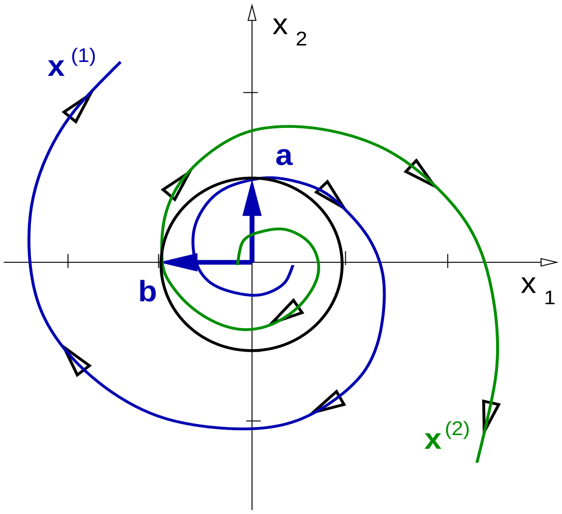
Solution:

The phase portrait of the solutions

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

$$\tilde{\mathbf{x}}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t},$$

are outgoing spirals.



Phase portraits for 2×2 systems.

Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}, \quad \mathbf{x}^{(2)} = \big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big]\, e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Solution: