Linear Systems of Differential Equations

Plan: * $n \times n$ Linear Differential Systems.

* Example: Reduction of order.
* Idea to solve 2×2 LDS: Decouple the System.
* $n \times n$ linear, Homogeneous, constant Coefficients.
* Solutions for $n \times n$ LHCC Diagonalizable Systems.

* $n \times n$ Linear Differential Systems.

Def: An first order $n \times n$ linear differential system is a differential equation of the form

$$\underline{x}'(t) = A(t)\,\underline{x}(t) + \underline{b}(t)$$

where $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is the unknown n-vector-valued function

while $A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}$ is the $n \times n$ coefficient matrix

and $\underline{b}(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}$ is the source n-vector

Recall: * matrix notation means:

$$\underline{x}' = A\underline{x} + \underline{b} \iff \begin{array}{l} x_1' = a_{11}\, x_1 + \cdots + a_{1n}\, x_n + b_1 \\ \vdots \\ x_n' = a_{n1}\, x_1 + \cdots + a_{nn}\, x_n + b_n \end{array}$$

* $$\underline{x}' = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$$

Example :

1) 1x1 System: Find x_1 solution of:

$$x_1' = a_{11}(t)\, x_1 + b_1 \iff y' = a(t)\, y + b(t)$$

$$x_1 = y, \quad a_{11} = a, \quad b_1 = b$$

2) 2x2 System: Find x_1, x_2 solutions of

$$x_1' = a_{11} x_1 + a_{12} x_2 + b_1$$

$$x_2' = a_{21} x_1 + a_{22} x_2 + b_2.$$

matrix notation:
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{(=A)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{(=\underline{x})} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{(=\underline{b})}$$

Find $\underline{x}$ solution of: $\underline{x}' = A\,\underline{x} + \underline{b}$

3) 3x3 System: Find x_1, x_2, x_3 solutions of:

$$x_1' = a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + b_1$$

$$x_2' = a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + b_2$$

$$x_3' = a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + b_3$$

* Idea to solve 2x2 LDS: Decouple the System.

Remark: The main difficulty to solve a 2x2 LDS is that the equations are coupled:

$$x_1' = a_{11} x_1 + a_{12} x_2 + b_1$$ ← equation for x_1, it also appears x_2

$$x_2' = a_{21} x_1 + a_{22} x_2 + b_2$$ ← equation for x_2 it also appears x_1.

Idea: Decouple the system.

Example: Find x_1, x_2 solutions of:

$$x_1' = x_1 - x_2$$
$$x_2' = -x_1 + x_2$$

Sol: We decouple the system:

Add up the two eqs: $(x_1 + x_2)' = 0$.

Subtract the second eq from the first: $(x_1 - x_2)' = 2x_1 - 2x_2$.

Introduce the functions: $u = x_1 + x_2$; $v = x_1 - x_2$

Then, the system is: $\boxed{u' = 0}$; $\boxed{v' = 2v}$

The equations are decoupled for u, v.

We solve each equation independently of the other.

$\boxed{u' = u_0}$; $\boxed{v' = v_0 e^{2t}}$

We now transform back to the original unknowns:

$$x_1 = \frac{u+v}{2} \Rightarrow x_1 = \frac{1}{2}\left(u_0 + v_0 e^{2t}\right)$$

$$x_2 = \frac{u-v}{2} \Rightarrow x_2 = \frac{1}{2}\left(u_0 - v_0 e^{2t}\right)$$

* $n\times n$ linear, Homogeneous, constant Coefficients.

Def: [The first order LDS $\underline{x}'(t) = A(t)\,\underline{x}(t) + \underline{b}(t)$
is called homogeneous iff holds $\underline{b} \equiv 0$;
is called of constant coefficients iff $A(t) = A_0$.]

Remark: From now on we concentrate on constant coefficients, homogeneous LDS, $\boxed{x'(t) = A\,\underline{x}(t)}$

Remark: We concentrate in the case that A is diagonalizable.

Review: On Diagonalizable matrices.

Def: [An $n\times n$ matrix A is diagonalizable iff there exist both an invertible matrix P and a diagonal matrix D such that
$$A = PDP^{-1}.$$]

Example: Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Sol: Take $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$

Matrix P is invertible, since $\det(P) = 1+1 = 2 \neq 0$.

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(Recall: $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $\det(M) = ad - bc$

then $M^{-1} = \dfrac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

Therefore: $$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\frac{1}{2}$$

$$= \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \Rightarrow \quad PDP^{-1} = A \quad \blacksquare$$

Remark: Given a diagonalizable matrix A,
How do we compute P, D such that $PDP^{-1} = A$?

Thrm: A matrix A is diagonalizable iff matrix A has a linearly independent set of n eigenvectors $\{\underline{v}^{(1)}, \dots, \underline{v}^{(n)}\}$ with corresponding eigenvectors $\{\lambda_1, \dots, \lambda_n\}$.
Furthermore, it holds: $P = [\underline{v}^{(1)}, \dots, \underline{v}^{(n)}]$, $D = [\lambda_1, \dots, \lambda_n]$.

Example: Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable by finding its eigenvalues and eigenvectors.

Sol: Eigenvalues: Find $\lambda \in R$ sol. of: $\det(A - \lambda \underline{I}) = 0$.

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 9 = 0 \Rightarrow \lambda_\pm - 1 = \pm 3 \Rightarrow \boxed{\begin{matrix} \lambda_+ = 4 \\ \lambda_- = -2 \end{matrix}}$$

Eigenvectors: For $\lambda = 4$, $\underline{v}$ is sol. of: $A\underline{v} = 4\underline{v}$; $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

$$(A - 4\underline{I})\underline{v} = \underline{0} \Rightarrow A - 4\underline{I} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$$

Gauss operations $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 - v_2 = 0$

$v_1 = v_2$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2$ choose: $v_2 = 1$

$$\boxed{\lambda = 4, \quad \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\lambda = -2 \Rightarrow A\underline{v} = -2\underline{v} \Rightarrow (A + 2\underline{I})\underline{v} = 0$

$$A + 2\underline{I} = \begin{bmatrix} 1+2 & 3 \\ 3 & 1+2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 + v_2 = 0 \qquad v_1 = -v_2.$$

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2$; choose $v_2 = 1$

$$\boxed{\lambda = -2, \quad \underline{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$\blacksquare$

* Solutions for $n\times n$ LHCC Diagonalizable Systems.

Thrm: If the $n\times n$ matrix A is diagonalizable with linearly independent set of eigenvectors $\{\underline{v}^{(1)},\ldots,\underline{v}^{(n)}\}$ and corresponding eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$, then every solution $\underline{x}$ of the linear differential system

$$\underline{x}'(t) = A\,\underline{x}(t)$$

is given by

$$\underline{x}(t) = c_1\,\underline{v}^{(1)}e^{\lambda_1 t} + \cdots + c_n\,\underline{v}^{(n)}e^{\lambda_n t}$$

Remark: Idea of the proof: Decouple the system using the eigenvalues and eigenvectors of matrix A.

Proof: A is diagonalizable. $\Rightarrow$ $A = P D P^{-1}$,

with: $P = [\underline{v}^{(1)},\ldots,\underline{v}^{(n)}]$; $D = \mathrm{diag}[\lambda_1,\ldots,\lambda_n]$.

Now: $x'(t) = A\,\underline{x}(t) \Rightarrow \underline{x}' = (PDP^{-1})\,\underline{x}$

$$= PD\,(P^{-1}\underline{x}) \Rightarrow (P^{-1}\underline{x})' = D\,(P^{-1}\underline{x})$$

Let $\underline{y} = P^{-1}\underline{x}$ ($\underline{y}$ is a l.c. of the $\underline{x}$ unknowns)

$$\underline{y}' = D\,\underline{y} \Rightarrow \left.\begin{matrix} y_1' = \lambda_1 y_1 \\ \vdots \\ y_n' = \lambda_n y_n \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y_1 = c_1\,e^{\lambda_1 t} \\ \vdots \\ y_n = c_n\,e^{\lambda_n t} \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} c_1\,e^{\lambda_1 t} \\ \vdots \\ c_n\,e^{\lambda_n t} \end{bmatrix}$$

Back to $\underline{x}$ unknowns:

$$\underline{y} = P^{-1}\underline{x} \Rightarrow x = P\,\underline{y}$$

$$\underline{x}(t) = [\underline{v}^{(1)},\ldots,\underline{v}^{(n)}]\begin{bmatrix} c_1\,e^{\lambda_1 t} \\ \vdots \\ c_n\,e^{\lambda_n t} \end{bmatrix} = c_1\,\underline{v}^{(1)}e^{\lambda_1 t} + \cdots + c_n\,\underline{v}^{(n)}e^{\lambda_n t}$$

$\blacksquare$

Example: Find all $\underline{x}$ solutions of

$$\underline{x}' = A\,\underline{x}, \qquad \text{with} \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

<u>Sol</u>: We use Thrm above to write down the solution.

We know: $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$; $\lambda_1 = 4$, $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = -2$, $\underline{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Hence: $$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}}$$ <u>Vector Form</u>

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{4t} - c_2 e^{-2t} \\ c_1 e^{4t} + c_2 e^{-2t} \end{bmatrix} \Rightarrow \begin{array}{l} \boxed{x_1(t) = c_1 e^{4t} - c_2 e^{-2t}} \\ \boxed{x_2(t) = c_1 e^{4t} + c_2 e^{-2t}} \end{array}$$

<u>component form</u>.

<u>Sol</u>: Using the idea of the proof of the Thrm.

$$\underline{x}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)' = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2}{2} \\ \frac{-x_1 + x_2}{2} \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = \frac{x_1 + x_2}{2} \\ y_2 = \frac{-x_1 + x_2}{2}. \end{array}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \left. \begin{array}{l} y_1' = 4 y_1 \\ y_2' = -2 y_2 \end{array} \right\} \Rightarrow \begin{array}{l} y_1 = c_1 e^{4t} \\ y_2 = c_2 e^{-2t} \end{array}$$

$$\underline{y}(t) = \begin{bmatrix} c_1 e^{4t} \\ c_2 e^{-2t} \end{bmatrix}. \qquad \underline{y} = P^{-1} \underline{x} \Rightarrow \underline{x} = P\,\underline{y} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \underline{x} = \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \end{bmatrix} \Rightarrow \boxed{\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{4t} - c_2 e^{-2t} \\ c_1 e^{4t} + c_2 e^{-2t} \end{bmatrix}}.$$

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