Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.

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Convolution of two functions.

Definition

The *convolution* of piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ given by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Remarks:

- $f * g$ is also called the generalized product of f and g.
- The definition of convolution of two functions also holds in the case that one of the functions is a generalized function, like Dirac's delta.

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \displaystyle\int_0^t e^{-\tau} \sin(t - \tau)\, d\tau$.

Integrate by parts twice: $\displaystyle\int_0^t e^{-\tau} \sin(t - \tau)\, d\tau =$

$$\Big[e^{-\tau} \cos(t - \tau)\Big]\Big|_0^t - \Big[e^{-\tau} \sin(t - \tau)\Big]\Big|_0^t - \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau,$$

$$2 \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau = \Big[e^{-\tau} \cos(t - \tau)\Big]\Big|_0^t - \Big[e^{-\tau} \sin(t - \tau)\Big]\Big|_0^t,$$

$$2(f * g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).$$

We conclude: $(f * g)(t) = \dfrac{1}{2}\big[e^{-t} + \sin(t) - \cos(t)\big]$. ◁

Convolution solutions (Sect. 4.5).

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Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions f, g, and h, hold:

(i) *Commutativity:* $f * g = g * f$;

(ii) *Associativity:* $f * (g * h) = (f * g) * h$;

(iii) *Distributivity:* $f * (g + h) = f * g + f * h$;

(iv) *Neutral element:* $f * 0 = 0$;

(v) *Identity element:* $f * \delta = f$.

Proof:

(v):

$$(f * \delta)(t) = \int_0^t f(\tau)\, \delta(t - \tau)\, d\tau = \int_0^t f(\tau)\, \delta(\tau - t)\, d\tau = f(t).$$

Properties of convolutions.

Proof:
(1): Commutativity: $f * g = g * f$.

The definition of convolution is,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Change the integration variable: $\hat{\tau} = t - \tau$, hence $d\hat{\tau} = -d\tau$,

$$(f * g)(t) = \int_t^0 f(t-\hat{\tau})\, g(\hat{\tau})(-1)\, d\hat{\tau}$$

$$(f * g)(t) = \int_0^t g(\hat{\tau})\, f(t-\hat{\tau})\, d\hat{\tau}$$

We conclude: $(f * g)(t) = (g * f)(t)$. □

Convolution solutions (Sect. 4.5).

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Laplace Transform of a convolution.

Theorem (Laplace Transform)

If f, g have well-defined Laplace Transforms $\mathcal{L}[f]$, $\mathcal{L}[g]$, then

$$\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Proof: The key step is to interchange two integrals. We start we the product of the Laplace transforms,

$$\mathcal{L}[f]\,\mathcal{L}[g] = \Big[\int_0^\infty e^{-st} f(t)\,dt\Big]\Big[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}\Big],$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \Big(\int_0^\infty e^{-st} f(t)\,dt\Big)\,d\tilde{t},$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$$

Laplace Transform of a convolution.

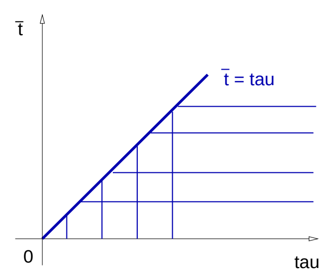
Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Change variables: $\tau = t + \tilde{t}$, hence $d\tau = dt$;

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \Big(\int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t})\,d\tau\Big)\,d\tilde{t}.$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_{\tilde{t}}^\infty e^{-s\tau} g(\tilde{t})\, f(\tau - \tilde{t})\,d\tau\,d\tilde{t}.$$

The key step: Switch the order of integration.



$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau} g(\tilde{t})\, f(\tau - \tilde{t})\,d\tilde{t}\,d\tau.$$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t}\, d\tau.$

Then, is straightforward to check that

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tau} \Big(\int_0^\tau g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t} \Big)\, d\tau,$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tau} (g * f)(\tau)\, d\tau$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \mathcal{L}[g * f]$$

We conclude: $\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$ □

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2 - 3)}.$$

Solution: We express F as a product of two Laplace Transforms,

$$F(s) = 3\,\frac{1}{s^3}\,\frac{1}{(s^2-3)} = \frac{3}{2}\,\frac{1}{\sqrt{3}} \Big(\frac{2}{s^3}\Big) \Big(\frac{\sqrt{3}}{s^2 - 3}\Big)$$

Recalling that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ and $\mathcal{L}[\sinh(at)] = \dfrac{a}{s^2 - a^2}$,

$$F(s) = \frac{\sqrt{3}}{2}\,\mathcal{L}[t^2]\,\mathcal{L}\big[\sinh(\sqrt{3}\,t)\big] = \frac{\sqrt{3}}{2}\,\mathcal{L}\big[t^2 * \sin(\sqrt{3}\,t)\big].$$

We conclude that $f(t) = \dfrac{\sqrt{3}}{2} \displaystyle\int_0^t \tau^2 \sinh\big[\sqrt{3}(t - \tau))\big]\, d\tau.$ ◁

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

$$f(t) = (g * h)(t), \qquad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$

Since $\mathcal{L}[(g * h)(t)] = \mathcal{L}[g(t)]\, \mathcal{L}[h(t)]$, then,

$$F(s) = \mathcal{L}\Big[\int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau\Big] = \mathcal{L}\big[e^{-3t}\big]\, \mathcal{L}\big[\cos(2t)\big].$$

We conclude that $F(s) = \dfrac{s}{(s+3)(s^2+4)}$. ◁

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\, \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\, G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$, and $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, then

$$\mathcal{L}[y(t)] = H(s)\, G(s) \quad \Rightarrow \quad y(t) = (h * g)(t).$$

Function h is simple to compute:

$$H(s) = \frac{1}{(s-2)(s-3)} = \frac{a}{(s-2)} + \frac{b}{(s-3)} = \frac{a(s-3) + b(s-2)}{(s-2)(s-3)}$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2 \quad \Rightarrow \quad a = -1. \qquad s = 3 \quad \Rightarrow \quad b = 1.$$

Therefore $H(s) = -\dfrac{1}{(s-2)} + \dfrac{1}{(s-3)}$. Then

$$h(t) = -e^{2t} + e^{3t}.$$

Recalling the formula $y(t) = (h * g)(t)$, we get

$$y(t) = \int_0^t \left(-e^{2\tau} + e^{3\tau}\right) g(t-\tau)\, d\tau. \qquad \triangleleft$$

Convolution solutions (Sect. 4.5).

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Impulse response solution.

Definition

The *impulse response solution* is the solution y_δ to the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{s^2 + a_1 s + a_0}\Big].$$

Denoting the characteristic polynomial by $p(s) = s^2 + a_1 s + a_0$,

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big].$$

Summary: The impulse reponse solution is the inverse Laplace Transform of the reciprocal of the equation characteristic polynomial.

Impulse response solution.

Recall: The impulse response solution is y_δ solution of the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y_{\delta_c}'' + 2\, y_{\delta_c}' + 2\, y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y_{\delta_c}'(0) = 0, \quad c \in \mathbb{R}.$$

Solution: $\mathcal{L}[y_{\delta_c}''] + 2\, \mathcal{L}[y_{\delta_c}'] + 2\, \mathcal{L}[y_{\delta_c}] = \mathcal{L}[\delta(t - c)].$

$$(s^2 + 2s + 2)\, \mathcal{L}[y_{\delta_c}] = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$$

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2+2s+2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4-8}\right]$$

Complex roots. We complete the square:

$$s^2 + 2s + 2 = \left[s^2 + 2\left(\frac{2}{2}\right)s + 1\right] - 1 + 2 = (s+1)^2 + 1.$$

Therefore, $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2+1}$.

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2+1}$.

Recall: $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2+1}$, and $\mathcal{L}[f](s-c) = \mathcal{L}[e^{ct}\,f(t)]$.

$$\frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = e^{-cs}\,\mathcal{L}[e^{-t}\,\sin(t)].$$

Since $e^{-cs}\,\mathcal{L}[f](s) = \mathcal{L}[u(t-c)\,f(t-c)]$,

we conclude $y_{\delta_c}(t) = u(t-c)\,e^{-(t-c)}\,\sin(t-c)$. $\triangleleft$

Convolution solutions (Sect. 4.5).

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Solution decomposition theorem.

Theorem (Solution decomposition)

The solution y to the IVP

$$y'' + a_1\, y' + a_0\, y = g(t), \quad y(0) = y_0, \quad y'(0) = y_1,$$

can be decomposed as

$$y(t) = y_h(t) + (y_\delta * g)(t),$$

where y_h is the solution of the homogeneous IVP

$$y_h'' + a_1\, y_h' + a_0\, y_h = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1,$$

and y_δ is the impulse response solution, that is,

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: $\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\sin(at)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,(1) - (-1), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - 1.$$

$$(s^2 + 2s + 2)\,\mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)].$$

$$\mathcal{L}[y] = \frac{(s+1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)}\,\mathcal{L}[\sin(at)].$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2 + 2s + 2)} + \dfrac{1}{(s^2 + 2s + 2)}\,\mathcal{L}[\sin(at)].$

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2 + 2s + 2)} = \dfrac{(s+1)}{(s+1)^2 + 1} = \mathcal{L}[e^{-t}\cos(t)],$

and: $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2 + 2s + 2)} = \dfrac{1}{(s+1)^2 + 1} = \mathcal{L}[e^{-t}\sin(t)].$ So,

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta * g)(t),$$

So: $y(t) = e^{-t}\cos(t) + \displaystyle\int_0^t e^{-\tau}\sin(\tau)\,\sin[a(t-\tau)]\,d\tau.$ $\triangleleft$

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\,\mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \dfrac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2 + a_1 s + a_0)}$.

Since, $\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)]$, so $y(t) = y_h(t) + (y_\delta * g)(t)$.

Equivalently: $y(t) = y_h(t) + \displaystyle\int_0^t y_\delta(\tau) g(t - \tau)\,d\tau$. □