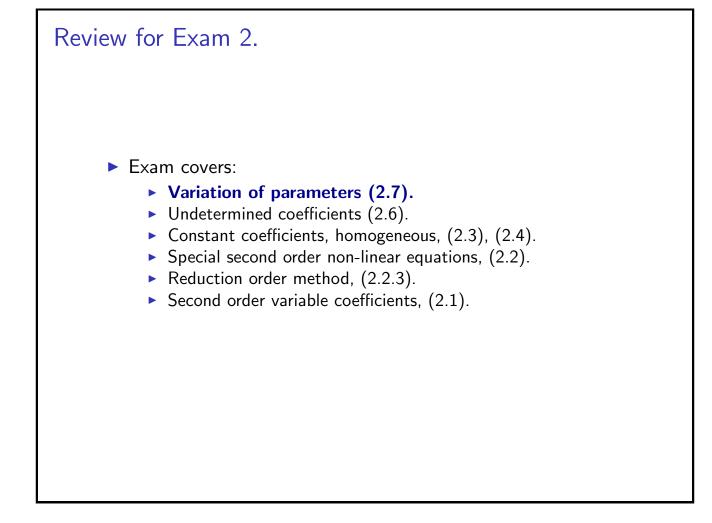
## Review for Exam 2.

- ▶ 6 problems, 60 minutes, in CC-415.
- ▶ 5 grading attempts per problem.
- Problems similar to homeworks.
- Integration table provided in handout.
- No notes, no books, no calculators.
- MLC mth 235 Exam 2 review handout: http://math.msu.edu/mlc/review-handouts/spring-13/mth235-e2.pdf
- Section 2.5 Applications is not covered.
- Exam covers:
  - ► Variation of parameters (2.7).
  - Undetermined coefficients (2.6).
  - Constant coefficients, homogeneous, (2.3), (2.4).
  - Special second order non-linear equations, (2.2).
  - ▶ Reduction order method, (2.2.3).
  - Second order variable coefficients, (2.1).

# Review for Exam 2. Notation for webwork: Consider the equation: $y'' + a_1 y' + a_2 y = 0.$ Let $r_*$ , $r_*$ be the roots of the characteristic polynomial. • If $r_* > r_*$ real, then • First fundamental solution: $y_1(t) = e^{r_*t}$ . • Second fundamental solution: $y_2(t) = e^{r_*t}$ . • If $r_{\pm} = \alpha \pm i\beta$ complex, then • First fundamental solution: $y_1(t) = e^{\alpha t} \cos(\beta t)$ . • Second fundamental solution: $y_2(t) = e^{\alpha t} \sin(\beta t)$ . • If $r_* = r_- = r$ real, then • First fundamental solution: $y_1(t) = e^{rt}$ . • Second fundamental solution: $y_1(t) = e^{rt}$ .



# Variation of parameters (2.7).

## Example

Find a particular solution of the equation

$$x^2 y'' - 6x y' + 10 y = 2x^{10},$$

knowing that  $y_1 = x^5$  and  $y_2 = x^2$  are solutions to the homogeneous equation.

Solution: We first need to divide the equation by  $x^2$ ,

$$y'' - \frac{6}{x}y' + \frac{10}{x^2}y = 2x^8,$$

Then the source function is  $f(x) = 2x^8$ . We now compute the Wronskian of  $y_1$ ,  $y_2$ ,

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} x^5 & x^2 \\ 5x^4 & 2x \end{vmatrix} = 2x^6 - 5x^6.$$

Hence  $W = -3x^6$ .

Variation of parameters (2.7).  
Example  
Find a particular solution of the equation  

$$x^2 y'' - 6x y' + 10 y = 2x^{10}$$
,  
knowing that  $y_1 = x^5$  and  $y_2 = x^2$  are solutions to the  
homogeneous equation.  
Solution:  $y_1 = x^5$ ,  $y_2 = x^2$ ,  $f = 2x^8$ ,  $W = -3x^6$ .  
Now we find the functions  $u_1$  and  $u_2$ ,  
 $u'_1 = -\frac{y_2 f}{W} = -\frac{x^2 2x^8}{(-3)x^6} = \frac{2}{3}x^4 \Rightarrow u_1 = \frac{2}{15}x^5$ .  
 $u'_2 = \frac{y_1 f}{W} = \frac{x^5 2x^8}{(-3)x^6} = -\frac{2}{3}x^7 \Rightarrow u_2 = -\frac{2}{24}x^8$ .  
 $y_p = u_1y_1 + u_2y_2 = \frac{2}{15}x^5x^5 - \frac{2}{24}x^8x^2 = \frac{2}{3}x^{10}(\frac{1}{5} - \frac{1}{8})$   
that is,  $y_p = \frac{2}{3}x^{10}(\frac{8-5}{40})$ , hence,  $y_p = \frac{1}{20}x^{10}$ .

# Variation of parameters (2.7).

### Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2} e^{-2x}.$$

Solution: We find the solutions of the homogeneous equation,

$$r^{2} + 4r + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[ -4 \pm \sqrt{16 - 16} \right] \quad \Rightarrow \quad r_{\pm} = -2.$$

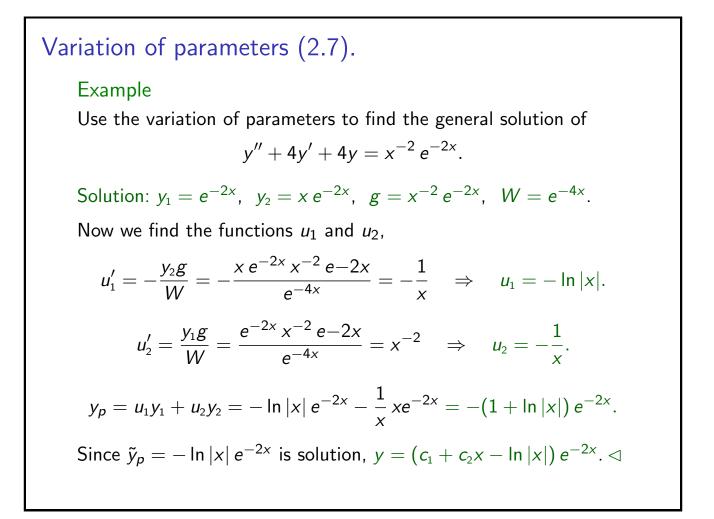
Fundamental solutions of the homogeneous equations are

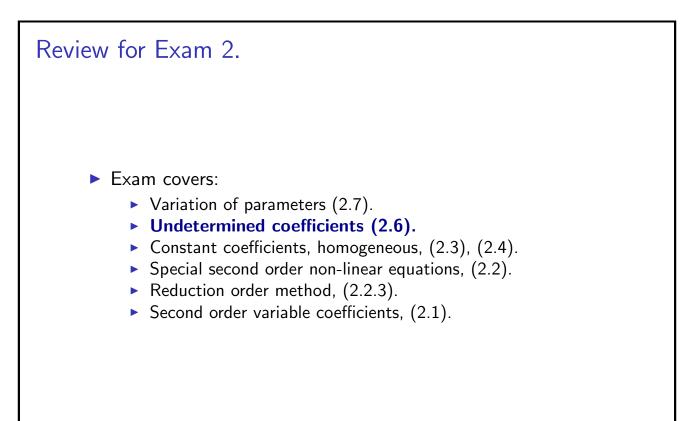
$$y_1 = e^{-2x}, \quad y_2 = x e^{-2x}.$$

We now compute their Wronskian,

 $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & (1-2x) e^{-2x} \end{vmatrix} = (1-2x) e^{-4x} + 2x e^{-4x}.$ 

Hence  $W = e^{-4x}$ .





Unde	etermined coefficien	ts (2.6).
G	uessing Solution Table.	
	$f_i(t)$ (K, m, a, b, given.)	$y_{p_i}(t)$ (Guess) (k not given.)
	Ke <sup>at</sup>	ke <sup>at</sup>
	Kt <sup>m</sup>	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
	K cos(bt)	$k_1\cos(bt) + k_2\sin(bt)$
	K sin(bt)	$k_1\cos(bt) + k_2\sin(bt)$
	Kt <sup>m</sup> e <sup>at</sup>	$e^{at}(k_mt^m+\cdots+k_0)$
	$Ke^{at}\cos(bt)$	$e^{at}[k_1\cos(bt)+k_2\sin(bt)]$
	KKe <sup>at</sup> sin(bt)	$e^{at}[k_1\cos(bt)+k_2\sin(bt)]$
	$Kt^m \cos(bt)$	$(k_m t^m + \cdots + k_0) [a_1 \cos(bt) + a_2 \sin(bt)]$
	Kt <sup>m</sup> sin(bt)	$(k_m t^m + \cdots + k_0) [a_1 \cos(bt) + a_2 \sin(bt)]$

## Example

Find all the solutions to the inhomogeneous equation

$$y^{\prime\prime}-3y^{\prime}-4y=2\sin(t).$$

Solution: We know that the general solution to homogeneous equation is  $y(t) = c_1 e^{4t} + c_2 e^{-t}$ .

Following the table: Since  $f = 2\sin(t)$ , then we guess

$$y_p = k_1 \sin(t) + k_2 \cos(t).$$

This guess satisfies  $L(y_p) \neq 0$ .

Compute: 
$$y'_p = k_1 \cos(t) - k_2 \sin(t)$$
,  $y''_p = -k_1 \sin(t) - k_2 \cos(t)$ .

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$
  
-4 $[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$ 

Undetermined coefficients (2.6).  
Example  
Find all the solutions to the inhomogeneous equation  

$$y'' - 3y' - 4y = 2\sin(t)$$
.  
Solution: Recall:  
 $L(y_p) = [-k_1\sin(t) - k_2\cos(t)] - 3[k_1\cos(t) - k_2\sin(t)] -4[k_1\sin(t) + k_2\cos(t)] = 2\sin(t),$   
 $(-5k_1 + 3k_2)\sin(t) + (-3k_1 - 5k_2)\cos(t) = 2\sin(t)$ .  
This equation holds for all  $t \in \mathbb{R}$ . In particular, at  $t = \frac{\pi}{2}$ ,  $t = 0$ .  
 $-5k_1 + 3k_2 = 2$ ,  
 $-3k_1 - 5k_2 = 0$ ,  $\Rightarrow \begin{cases} k_1 = -\frac{5}{17}, \\ k_2 = \frac{3}{17}. \end{cases}$ 

## Example

Find all the solutions to the inhomogeneous equation

$$y^{\prime\prime}-3y^{\prime}-4y=2\sin(t).$$

Solution: Recall:  $k_1 = -\frac{5}{17}$  and  $k_2 = \frac{3}{17}$ .

So the particular solution to the inhomogeneous equation is

$$y_p(t) = \frac{1}{17} \left[ -5\sin(t) + 3\cos(t) \right].$$

The general solution is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17} \left[ -5\sin(t) + 3\cos(t) \right].$$

## Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}$$

Solution: Find the solutions of the homogeneous problem,

$$r^2 + 4 = 0 \implies r_{\pm} = \pm 2i.$$
  
 $y_1 = \cos(2x), \quad y_2 = \sin(2x).$ 

Start with the first source,  $f_1(x) = 3\sin(2x)$ . The function  $\tilde{y}_{p_1} = k_1\sin(2x) + k_2\cos(2x)$  is the wrong guess, since it is solution of the homogeneous equation. We guess:

 $y_p = x \big[ k_1 \sin(2x) + k_2 \cos(2x) \big].$ 

$$y'_{p} = [k_{1}\sin(2x) + k_{2}\cos(2x)] + 2x[k_{1}\cos(2x) - k_{2}\sin(2x)].$$
  
$$y''_{p} = 4[k_{1}\cos(2x) - k_{2}\sin(2x)] + 4x[-k_{1}\sin(2x) - k_{2}\cos(2x)]$$

# Undetermined coefficients (2.6)

### Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}.$$

Solution: Recall:  $y_1 = \sin(2x)$ , and  $y_2 = \cos(2x)$ .

$$4[k_1\cos(2x) - k_2\sin(2x)] + 4x[-k_1\sin(2x) - k_2\cos(2x)] + 4x[k_1\sin(2x) + k_2\cos(2x)] = 3\sin(2x),$$

Therefore,  $4[k_1 \cos(2x) - k_2 \sin(2x)] = 3\sin(2x)$ . Evaluating at x = 0 and  $x = \pi/4$  we get

$$4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}$$
  
Therefore,  $y_{p_1} = -\frac{3}{4} x \cos(2x)$ .

# Undetermined coefficients (2.6) Example Use the undetermined coefficients to find the general solution of $y'' + 4y = 3\sin(2x) + e^{3x}$ . Solution: Recall: $y_{p_1} = -\frac{3}{4}x\cos(2x)$ . We now compute $y_{p_2}$ for $f_2(x) = e^{3x}$ . We guess: $y_{p_2} = k e^{3x}$ . Then, $y''_{p_2} = 9 e^{3x}$ , $(9+4)ke^{3x} = e^{3x} \Rightarrow k = \frac{1}{13} \Rightarrow y_{p_2} = \frac{1}{13}e^{3x}$ . Therefore, the general solution is $y(x) = c_1\sin(2x) + (c_2 - \frac{3}{4}x)\cos(2x) + \frac{1}{13}e^{3x}$ .

Undetermined coefficients (2.6). Example • For  $y'' - 3y' - 4y = 3e^{2t} \sin(t)$ , guess  $y_p(t) = [k_1 \sin(t) + k_2 \cos(t)] e^{2t}$ . • For  $y'' - 3y' - 4y = 2t^2 e^{3t}$ , guess  $y_p(t) = (k_0 + k_1 t + k_2 t^2) e^{3t}$ . • For  $y'' - 3y' - 4y = 3t \sin(t)$ , guess  $y_p(t) = (1 + k_1 t) [k_2 \sin(t) + k_3 \cos(t)]$ .

### Example

Find a particular solution to

$$y'' + 2y' - 2y = e^{-4it}.$$

Using this solution find particular solutions to the equations

$$y'' + 2y' - 2y = \cos(-4t), \qquad y'' + 2y' - 2y = \sin(-4t).$$

Solution: Since the source is and exponential  $f(t) = e^{-4it}$ , we guess as particular solution the exponential  $y_p(t) = k e^{-4it}$ . We now check whether  $y_p$  is solution of the homogeneous eq.:

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4+8} \right] \quad \Rightarrow \quad \text{Real roots.}$$

Hence  $y_p$  is not solution of the homogeneous equation.

## Undetermined coefficients (2.6).

### Example

Find a particular solution to

$$y'' + 2y' - 2y = e^{-4it}.$$

Using this solution find particular solutions to the equations

$$y'' + 2y' - 2y = \cos(-4t), \qquad y'' + 2y' - 2y = \sin(-4t).$$

Solution: Recall:  $y_p(t) = k e^{-4it}$ .

$$[(-4i)^2 + 2(-4i) - 2]ke^{-4it} = e^{-4it} \Rightarrow (-16 - 8i - 2)k = 1$$

$$k = -rac{1}{18+8i} = -rac{1}{2} rac{1}{(9+4i)} rac{(9-4i)}{(9-4i)} = -rac{1}{2} rac{(9-4i)}{(9^2+4^2)}.$$

Hence,  $y_p(t) = -\frac{1}{2(9^2 + 4^2)} (9 - 4i) e^{-4it}$ .

## Example

Find a particular solution to

$$y'' + 2y' - 2y = e^{-4it}.$$

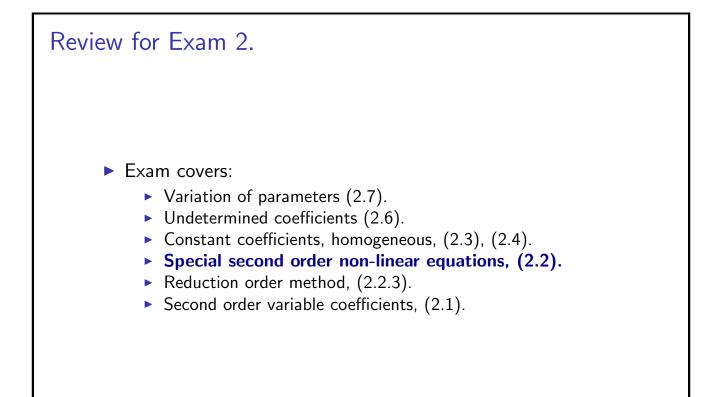
Using this solution find particular solutions to the equations

$$y'' + 2y' - 2y = \cos(-4t), \qquad y'' + 2y' - 2y = \sin(-4t)$$

Solution: Recall:  $y_p(t) = -\frac{1}{2(9^2 + 4^2)} (9 - 4i) e^{-4it}$ .

For the second part of the problem, we need to compute the real and imaginary parts of or solution:

$$y_{p}(t) = -\frac{1}{2(9^{2} + 4^{2})}(9 - 4i)\left[\cos(4t) - i\sin(4t)\right]$$
$$y_{p_{r}} = -\frac{1}{2(9^{2} + 4^{2})}\left[9\cos(4t) - 4\sin(4t)\right]$$
$$y_{p_{i}} = -\frac{1}{2(9^{2} + 4^{2})}\left[-4\cos(4t) - 9\sin(4t)\right]$$



Special second order non-linear equations, (2.2).

## Example

Find the solution y of the IVP

$$y y'' + 4(y')^2 = 0,$$
  $y(0) = 1,$   $y'(0) = 7.$ 

Solution: This is an equation of the form y'' = f(y, y'), (t missing). Introduce the function v(t) = y'(t), that implies v'(t) = y''(t), so

$$yv'+4v^2=0 \quad \Rightarrow \quad v'=-4\frac{v^2}{y}, \qquad v(0)=7.$$

The difficulty is that y still appears in the equation. We now look only for invertible solutions functions  $t \mapsto y(t)$ , that is, we have the inverse function  $y \mapsto t(y)$ . For this type of solutions, introduce the function

$$w(y) = v(t(y)).$$

Special second order non-linear equations, (2.2).

## Example

Find the solution y of the IVP

$$y y'' + 4(y')^2 = 0,$$
  $y(0) = 1,$   $y'(0) = 7.$ 

Solution: Recall:  $v' = -4 \frac{v^2}{y}$ , with v(0) = 7, and w(y) = v(t(y)). The initial conditions for w are obtained as follows:

$$y(t=0)=1 \quad \Leftrightarrow \quad t(y=1)=0,$$

$$w(y = 1) = v(t(y = 1)) = v(0) = 7 \Rightarrow w(1) = 7.$$

Chain rule on *w* always implies the equation:

$$w'(y) = rac{v'(t(y))}{w(y)} \quad \Rightarrow \quad w' = -4rac{w^2}{y}rac{1}{w} = -4rac{w}{y}$$

Special second order non-linear equations, (2.2). Example Find the solution y of the IVP  $y y'' + 4(y')^2 = 0$ , y(0) = 1, y'(0) = 7. Solution: Recall:  $w' = -4 \frac{w}{y}$ , with w(1) = 7, and w(y) = v(t(y)).  $\frac{w'}{w} = -\frac{4}{y} \Rightarrow \ln(w) = -4\ln(y) + c = \ln(y^{-4}) + c$ We obtain the solution  $w(y) = \tilde{c} y^{-4}$ . The initial condition implies  $7 = w(1) = \tilde{c}$ , hence  $w(y) = 7 y^{-4}$ . We now consider y as function of t, and we recall that  $w(y) = v(t(y)) \Leftrightarrow w(y(t)) = v(t) = y'(t)$ .  $\frac{7}{y^4(t)} = w(y(t)) = v(t) = y'(t) \Rightarrow y'(t) = \frac{7}{y^4(t)}$ .

Special second order non-linear equations, (2.2).

Example

Find the solution y of the IVP

$$y y'' + 4(y')^2 = 0,$$
  $y(0) = 1,$   $y'(0) = 7.$ 

Solution: Recall:  $y'(t) = \frac{7}{y^4(t)}$ , with y(0) = 1.

$$y^4 y' = 7 \quad \Rightarrow \quad \frac{y^5}{5} = 7t + c.$$

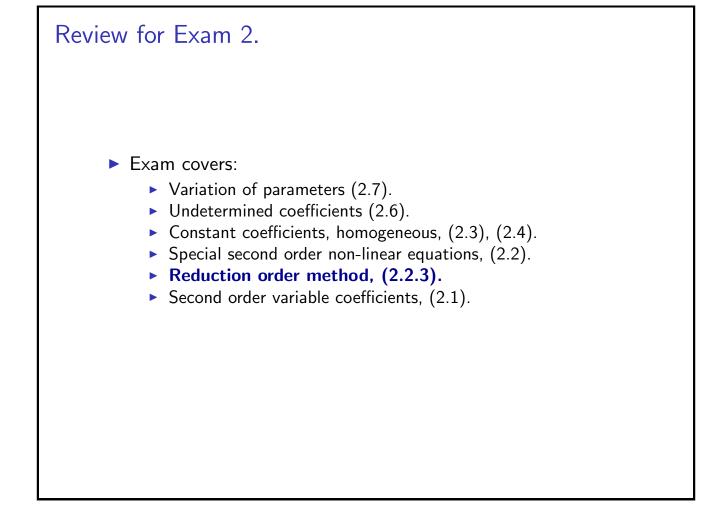
The initial condition fixes the integration constant,

$$\frac{1}{5}=c, \quad \Rightarrow \quad \frac{y^5(t)}{5}=7t+\frac{1}{5}.$$

We then obtain the solution of the  $\ensuremath{\mathsf{IVP}}$  as

$$y(t)=\sqrt[5]{35\,t+1}.$$

 $\triangleleft$ 



## Reduction order method, (2.2.3).

### Example

Find a fundamental set of solutions to

$$t^2y'' + 2ty' - 2y = 0,$$

knowing that  $y_1(t) = t$  is a solution.

Solution: Express  $y_2(t) = v(t) y_1(t)$ . The equation for v comes from  $t^2 y_2'' + 2ty_2' - 2y_2 = 0$ . We need to compute

$$y_2 = v t,$$
  $y'_2 = t v' + v,$   $y''_2 = t v'' + 2v'.$ 

So, the equation for v is given by

$$t^{2}(t v'' + 2v') + 2t(t v' + v) - 2t v = 0$$
  

$$t^{3} v'' + (2t^{2} + 2t^{2}) v' + (2t - 2t) v = 0$$
  

$$t^{3} v'' + (4t^{2}) v' = 0 \implies v'' + \frac{4}{t} v' = 0.$$

Reduction order method, (2.2.3). Example Find a fundamental set of solutions to  $t^2y'' + 2ty' - 2y = 0$ , knowing that  $y_1(t) = t$  is a solution. Solution: Recall:  $v'' + \frac{4}{t}v' = 0$ . This is a first order equation for w = v', given by  $w' + \frac{4}{t}w = 0$ , so  $\frac{w'}{w} = -\frac{4}{t} \Rightarrow \ln(w) = -4\ln(t) + c_0 \Rightarrow w(t) = c_1t^{-4}, c_1 \in \mathbb{R}$ . Integrating w we obtain v, that is,  $v = c_2t^{-3} + c_3$ , with  $c_2, c_3 \in \mathbb{R}$ . Recalling that  $y_2 = t v$  we then conclude that  $y_2 = c_2t^{-2} + c_3t$ . Choosing  $c_2 = 1$  and  $c_3 = 0$  we obtain the fundamental solutions  $y_1(t) = t$  and  $y_2(t) = \frac{1}{t^2}$ .

Review for Exam 2.
Exam covers:

Variation of parameters (2.7).
Undetermined coefficients (2.6).
Constant coefficients, homogeneous, (2.3), (2.4).
Special second order non-linear equations, (2.2).
Reduction order method, (2.2.3).

Second order variable coefficients, (2.1).

Second order variable coefficients, (2.1). Example Find the Wronskian of two solutions of the equation  $t^2 y'' - t(t+2) y' + (t+2) y = 0, \quad t > 0.$ Solution: Write the equation as in Abel's Theorem,  $y'' - \left(\frac{2}{t} + 1\right) y' + \left(\frac{2}{t^2} + \frac{1}{t}\right) y = 0.$ Abel's Theorem says that the Wronskian satisfies the equation  $W'_{y_1y_2}(t) - \left(\frac{2}{t} + 1\right) W_{y_1y_2}(t) = 0.$ This is a first order, linear equation for  $W_{y_1y_2}$ . The integrating factor method implies  $A(t) = -\int_{t_0}^t \left(\frac{2}{s} + 1\right) ds = -2 \ln\left(\frac{t}{t_0}\right) - (t - t_0)$ 

Second order variable coefficients, (2.1).

### Example

Find the Wronskian of two solutions of the equation

$$t^2 y'' - t(t+2) y' + (t+2) y = 0, \qquad t > 0.$$

Solution:  $A(t) = -2 \ln\left(\frac{t}{t_0}\right) - (t - t_0) = \ln\left(\frac{t_0^2}{t^2}\right) - (t - t_0).$ 

The integrating factor is  $\mu = rac{t_0^2}{t^2} e^{-(t-t_0)}$ . Therefore,

$$\Big[\mu(t)W_{y_1y_2}(t)\Big]'=0 \quad \Rightarrow \quad \mu(t)W_{y_1y_2}(t)-\mu(t_0)W_{y_1y_2}(t_0)=0$$

so, the solution is  $W_{y_1y_2}(t) = W_{y_1y_2}(t_0) \frac{t^2}{t_0^2} e^{(t-t_0)}$ . Denoting  $c = (W_{y_1y_2}(t_0)/t_0^2) e^{-t_0}$ , then  $W_{y_1y_2}(t) = c t^2 e^t$ .