Review for Exam 2.

- 6 problems, 60 minutes, in CC-415.
- 5 grading attempts per problem.
- Problems similar to homeworks.
- Integration table provided in handout.
- No notes, no books, no calculators.
- MLC mth 235 Exam 2 review handout:
  `http://math.msu.edu/mlc/review-handouts/spring-13/mth235-e2.pdf`
- Section 2.5 Applications is not covered.
- Exam covers:
  - Variation of parameters (2.7).
  - Undetermined coefficients (2.6).
  - Constant coefficients, homogeneous, (2.3), (2.4).
  - Special second order non-linear equations, (2.2).
  - Reduction order method, (2.2.3).
  - Second order variable coefficients, (2.1).

Review for Exam 2.

Notation for webwork: Consider the equation:

$$y'' + a_1\, y' + a_2\, y = 0.$$

Let r_+, r_- be the roots of the characteristic polynomial.

- If $r_+ > r_-$ real, then
  - First fundamental solution: $y_1(t) = e^{r_+ t}$.
  - Second fundamental solution: $y_2(t) = e^{r_- t}$.
- If $r_\pm = \alpha \pm i\beta$ complex, then
  - First fundamental solution: $y_1(t) = e^{\alpha t} \cos(\beta t)$.
  - Second fundamental solution: $y_2(t) = e^{\alpha t} \sin(\beta t)$.
- If $r_+ = r_- = r$ real, then
  - First fundamental solution: $y_1(t) = e^{rt}$.
  - Second fundamental solution: $y_2(t) = t\, e^{rt}$.

Review for Exam 2.

- Exam covers:
  - **Variation of parameters (2.7).**
  - Undetermined coefficients (2.6).
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  - Reduction order method, (2.2.3).
  - Second order variable coefficients, (2.1).

Variation of parameters (2.7).

Example

Find a particular solution of the equation

$$x^2\, y'' - 6x\, y' + 10\, y = 2x^{10},$$

knowing that $y_1 = x^5$ and $y_2 = x^2$ are solutions to the homogeneous equation.

Solution: We first need to divide the equation by x^2,

$$y'' - \frac{6}{x}\, y' + \frac{10}{x^2}\, y = 2x^8,$$

Then the source function is $f(x) = 2x^8$. We now compute the Wronskian of y_1, y_2,,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^5 & x^2 \\ 5x^4 & 2x \end{vmatrix} = 2x^6 - 5x^6.$$

Hence $W = -3x^6$.

Variation of parameters (2.7).

Example

Find a particular solution of the equation

$$x^2\, y'' - 6x\, y' + 10\, y = 2x^{10},$$

knowing that $y_1 = x^5$ and $y_2 = x^2$ are solutions to the homogeneous equation.

Solution: $y_1 = x^5, \quad y_2 = x^2, \quad f = 2x^8, \quad W = -3x^6.$

Now we find the functions u_1 and u_2,

$$u_1' = -\frac{y_2 f}{W} = -\frac{x^2\, 2x^8}{(-3)x^6} = \frac{2}{3}\, x^4 \quad \Rightarrow \quad u_1 = \frac{2}{15}\, x^5.$$

$$u_2' = \frac{y_1 f}{W} = \frac{x^5 2x^8}{(-3)x^6} = -\frac{2}{3} x^7 \quad \Rightarrow \quad u_2 = -\frac{2}{24}\, x^8.$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{2}{15}\, x^5\, x^5 - \frac{2}{24}\, x^8\, x^2 = \frac{2}{3}\, x^{10} \Big(\frac{1}{5} - \frac{1}{8} \Big)$$

that is, $y_p = \frac{2}{3}\, x^{10} \Big(\frac{8-5}{40} \Big)$, hence, $y_p = \frac{1}{20}\, x^{10}$. $\lhd$

Variation of parameters (2.7).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

Solution: We find the solutions of the homogeneous equation,

$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-4 \pm \sqrt{16 - 16}\right] \quad \Rightarrow \quad r_\pm = -2.$$

Fundamental solutions of the homogeneous equations are

$$y_1 = e^{-2x}, \quad y_2 = x\, e^{-2x}.$$

We now compute their Wronskian,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & x\, e^{-2x} \\ -2e^{-2x} & (1-2x)\, e^{-2x} \end{vmatrix} = (1-2x)\, e^{-4x} + 2x\, e^{-4x}.$$

Hence $W = e^{-4x}$.

Variation of parameters (2.7).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

Solution: $y_1 = e^{-2x}$, $y_2 = x\, e^{-2x}$, $g = x^{-2}\, e^{-2x}$, $W = e^{-4x}$.

Now we find the functions u_1 and u_2,

$$u_1' = -\frac{y_2 g}{W} = -\frac{x\, e^{-2x}\, x^{-2}\, e{-}2x}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|.$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2x}\, x^{-2}\, e{-}2x}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = -\frac{1}{x}.$$

$$y_p = u_1 y_1 + u_2 y_2 = -\ln|x|\, e^{-2x} - \frac{1}{x}\, xe^{-2x} = -(1 + \ln|x|)\, e^{-2x}.$$

Since $\tilde{y}_p = -\ln|x|\, e^{-2x}$ is solution, $y = (c_1 + c_2 x - \ln|x|)\, e^{-2x}$. $\lhd$

Review for Exam 2.

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  - Constant coefficients, homogeneous, (2.3), (2.4).
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Undetermined coefficients (2.6).

Guessing Solution Table.

$f_i(t)$ (K, m, a, b, given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$K\sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at}\cos(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$KKe^{at}\sin(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$Kt^m \cos(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$
$Kt^m \sin(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$

Undetermined coefficients (2.6).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: We know that the general solution to homogeneous equation is $y(t) = c_1 e^{4t} + c_2 e^{-t}$.

Following the table: Since $f = 2\sin(t)$, then we guess

$$y_p = k_1 \sin(t) + k_2 \cos(t).$$

This guess satisfies $L(y_p) \neq 0$.

Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$

$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

Undetermined coefficients (2.6).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall:

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$
$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

$$(-5k_1 + 3k_2)\sin(t) + (-3k_1 - 5k_2)\cos(t) = 2\sin(t).$$

This equation holds for all $t \in \mathbb{R}$. In particular, at $t = \frac{\pi}{2}$, $t = 0$.

$$\left.\begin{aligned} -5k_1 + 3k_2 = 2, \\ -3k_1 - 5k_2 = 0, \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} k_1 = -\dfrac{5}{17}, \\ k_2 = \dfrac{3}{17}. \end{cases}$$

Undetermined coefficients (2.6).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall: $k_1 = -\frac{5}{17}$ and $k_2 = \frac{3}{17}$.

So the particular solution to the inhomogeneous equation is

$$y_p(t) = \frac{1}{17}\big[-5\sin(t) + 3\cos(t)\big].$$

The general solution is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17}\big[-5\sin(t) + 3\cos(t)\big]. \qquad \triangleleft$$

Undetermined coefficients (2.6)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}$$

Solution: Find the solutions of the homogeneous problem,

$$r^2 + 4 = 0 \quad \Rightarrow \quad r_\pm = \pm 2i.$$

$$y_1 = \cos(2x), \quad y_2 = \sin(2x).$$

Start with the first source, $f_1(x) = 3\sin(2x)$.
The function $\tilde{y}_{p_1} = k_1 \sin(2x) + k_2 \cos(2x)$ is the wrong guess, since it is solution of the homogeneous equation. We guess:

$$y_p = x\big[k_1 \sin(2x) + k_2 \cos(2x)\big].$$

$$y_p' = \big[k_1 \sin(2x) + k_2 \cos(2x)\big] + 2x\big[k_1 \cos(2x) - k_2 \sin(2x)\big].$$

$$y_p'' = 4\big[k_1 \cos(2x) - k_2 \sin(2x)\big] + 4x\big[-k_1 \sin(2x) - k_2 \cos(2x)\big].$$

Undetermined coefficients (2.6)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}.$$

Solution: Recall: $y_1 = \sin(2x)$, and $y_2 = \cos(2x)$.

$$4\big[k_1 \cos(2x) - k_2 \sin(2x)\big] + 4x\big[-k_1 \sin(2x) - k_2 \cos(2x)\big] + 4x\big[k_1 \sin(2x) + k_2 \cos(2x)\big] = 3\sin(2x),$$

Therefore, $4\big[k_1 \cos(2x) - k_2 \sin(2x)\big] = 3\sin(2x)$.

Evaluating at $x = 0$ and $x = \pi/4$ we get

$$4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.$$

Therefore, $y_{p_1} = -\dfrac{3}{4}\, x \cos(2x)$.

Undetermined coefficients (2.6)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}.$$

Solution: Recall: $y_{p_1} = -\dfrac{3}{4}\, x\cos(2x)$.

We now compute y_{p_2} for $f_2(x) = e^{3x}$.

We guess: $y_{p_2} = k\, e^{3x}$. Then, $y''_{p_2} = 9\, e^{3x}$,

$$(9+4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13} \quad \Rightarrow \quad y_{p_2} = \frac{1}{13}\, e^{3x}.$$

Therefore, the general solution is

$$y(x) = c_1 \sin(2x) + \Big(c_2 - \frac{3}{4}\, x\Big) \cos(2x) + \frac{1}{13}\, e^{3x}. \qquad \triangleleft$$

Undetermined coefficients (2.6).

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \big[k_1 \sin(t) + k_2 \cos(t)\big]\, e^{2t}.$$

- For $y'' - 3y' - 4y = 2t^2\, e^{3t}$, guess

$$y_p(t) = \big(k_0 + k_1 t + k_2 t^2\big)\, e^{3t}.$$

- For $y'' - 3y' - 4y = 3t\, \sin(t)$, guess

$$y_p(t) = (1 + k_1 t)\, \big[k_2 \sin(t) + k_3 \cos(t)\big].$$

Undetermined coefficients (2.6).

Example

Find a particular solution to

$$y'' + 2y' - 2y = e^{-4it}.$$

Using this solution find particular solutions to the equations

$$y'' + 2y' - 2y = \cos(-4t), \qquad y'' + 2y' - 2y = \sin(-4t).$$

Solution: Since the source is and exponential $f(t) = e^{-4it}$, we guess as particular solution the exponential $y_p(t) = k\, e^{-4it}$.
We now check whether y_p is solution ot the homogeneous eq.:

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}\big[-2 \pm \sqrt{4+8}\big] \quad \Rightarrow \quad \text{Real roots.}$$

Hence y_p is not solution of the homogeneous equation.

Undetermined coefficients (2.6).

Example

Find a particular solution to

$$y'' + 2y' - 2y = e^{-4it}.$$

Using this solution find particular solutions to the equations

$$y'' + 2y' - 2y = \cos(-4t), \qquad y'' + 2y' - 2y = \sin(-4t).$$

Solution: Recall: $y_p(t) = k\, e^{-4it}$.

$$\big[(-4i)^2 + 2(-4i) - 2\big]\, k e^{-4it} = e^{-4it} \quad \Rightarrow \quad (-16 - 8i - 2)k = 1$$

$$k = -\frac{1}{18+8i} = -\frac{1}{2}\,\frac{1}{(9+4i)}\,\frac{(9-4i)}{(9-4i)} = -\frac{1}{2}\,\frac{(9-4i)}{(9^2+4^2)}.$$

Hence, $y_p(t) = -\dfrac{1}{2(9^2+4^2)}\,(9-4i)\, e^{-4it}$.

Undetermined coefficients (2.6).

Example

Find a particular solution to

$$y'' + 2y' - 2y = e^{-4it}.$$

Using this solution find particular solutions to the equations

$$y'' + 2y' - 2y = \cos(-4t), \qquad y'' + 2y' - 2y = \sin(-4t).$$

Solution: Recall: $y_p(t) = -\dfrac{1}{2(9^2+4^2)}\,(9-4i)\,e^{-4it}$.

For the second part of the problem, we need to compute the real and imaginary parts of or solution:

$$y_p(t) = -\frac{1}{2(9^2+4^2)}(9-4i)\big[\cos(4t) - i\sin(4t)\big]$$

$$y_{p_r} = -\frac{1}{2(9^2+4^2)}\,\big[9\cos(4t) - 4\sin(4t)\big]$$

$$y_{p_i} = -\frac{1}{2(9^2+4^2)}\,\big[-4\cos(4t) - 9\sin(4t)\big]$$

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  - **Special second order non-linear equations, (2.2).**
  - Reduction order method, (2.2.3).
  - Second order variable coefficients, (2.1).

Special second order non-linear equations, (2.2).

Example

Find the solution y of the IVP

$$y\,y'' + 4(y')^2 = 0, \qquad y(0) = 1, \quad y'(0) = 7.$$

Solution: This is an equation of the form $y'' = f(y, y')$, (t missing). Introduce the function $v(t) = y'(t)$, that implies $v'(t) = y''(t)$, so

$$yv' + 4v^2 = 0 \quad \Rightarrow \quad v' = -4\,\frac{v^2}{y}, \qquad v(0) = 7.$$

The difficulty is that y still appears in the equation.
We now look only for invertible solutions functions $t \mapsto y(t)$, that is, we have the inverse function $y \mapsto t(y)$.
For this type of solutions, introduce the function

$$w(y) = v(t(y)).$$

Special second order non-linear equations, (2.2).

Example

Find the solution y of the IVP

$$y\,y'' + 4(y')^2 = 0, \qquad y(0) = 1, \quad y'(0) = 7.$$

Solution: Recall: $v' = -4\,\frac{v^2}{y}$, with $v(0) = 7$, and $w(y) = v(t(y))$.
The initial conditions for w are obtained as follows:

$$y(t=0) = 1 \quad \Leftrightarrow \quad t(y=1) = 0,$$

$$w(y=1) = v(t(y=1)) = v(0) = 7 \quad \Rightarrow \quad w(1) = 7.$$

Chain rule on w always implies the equation:

$$w'(y) = \frac{v'(t(y))}{w(y)} \quad \Rightarrow \quad w' = -4\frac{w^2}{y}\,\frac{1}{w} = -4\frac{w}{y}.$$

Special second order non-linear equations, (2.2).

Example

Find the solution y of the IVP

$$y\,y'' + 4(y')^2 = 0, \qquad y(0) = 1, \quad y'(0) = 7.$$

Solution: Recall: $w' = -4\,\dfrac{w}{y}$, with $w(1) = 7$, and $w(y) = v(t(y))$.

$$\frac{w'}{w} = -\frac{4}{y} \;\Rightarrow\; \ln(w) = -4\ln(y) + c = \ln(y^{-4}) + c$$

We obtain the solution $w(y) = \tilde{c}\,y^{-4}$.
The initial condition implies $7 = w(1) = \tilde{c}$, hence $w(y) = 7\,y^{-4}$.
We now consider y as function of t, and we recall that

$$w(y) = v(t(y)) \quad \Leftrightarrow \quad w(y(t)) = v(t) = y'(t).$$

$$\frac{7}{y^4(t)} = w(y(t)) = v(t) = y'(t) \quad \Rightarrow \quad y'(t) = \frac{7}{y^4(t)}.$$

Special second order non-linear equations, (2.2).

Example

Find the solution y of the IVP

$$y\,y'' + 4(y')^2 = 0, \qquad y(0) = 1, \quad y'(0) = 7.$$

Solution: Recall: $y'(t) = \dfrac{7}{y^4(t)}$, with $y(0) = 1$.

$$y^4\,y' = 7 \quad \Rightarrow \quad \frac{y^5}{5} = 7t + c.$$

The initial condition fixes the integration constant,

$$\frac{1}{5} = c, \quad \Rightarrow \quad \frac{y^5(t)}{5} = 7t + \frac{1}{5}.$$

We then obtain the solution of the IVP as

$$y(t) = \sqrt[5]{35\,t + 1}.$$

$\triangleleft$

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  - **Reduction order method, (2.2.3).**
  - Second order variable coefficients, (2.1).

Reduction order method, (2.2.3).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2ty_2' - 2y_2 = 0$. We need to compute

$$y_2 = v\,t, \qquad y_2' = t\,v' + v, \qquad y_2'' = t\,v'' + 2v'.$$

So, the equation for v is given by

$$t^2\big(t\,v'' + 2v'\big) + 2t\big(t\,v' + v\big) - 2t\,v = 0$$

$$t^3\,v'' + (2t^2 + 2t^2)\,v' + (2t - 2t)\,v = 0$$

$$t^3\,v'' + (4t^2)\,v' = 0 \quad \Rightarrow \quad v'' + \frac{4}{t}v' = 0.$$

Reduction order method, (2.2.3).

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Recall: $v'' + \frac{4}{t} v' = 0$.

This is a first order equation for $w = v'$, given by $w' + \frac{4}{t} w = 0$, so

$$\frac{w'}{w} = -\frac{4}{t} \;\Rightarrow\; \ln(w) = -4\ln(t) + c_0 \;\Rightarrow\; w(t) = c_1 t^{-4}, \;\; c_1 \in \mathbb{R}.$$

Integrating w we obtain v, that is, $v = c_2 t^{-3} + c_3$, with $c_2, c_3 \in \mathbb{R}$.
Recalling that $y_2 = t\, v$ we then conclude that $y_2 = c_2 t^{-2} + c_3 t$.
Choosing $c_2 = 1$ and $c_3 = 0$ we obtain the fundamental solutions
$y_1(t) = t$ and $y_2(t) = \frac{1}{t^2}$. $\triangleleft$

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Second order variable coefficients, (2.1).

Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: Write the equation as in Abel's Theorem,

$$y'' - \Big(\frac{2}{t} + 1\Big)\, y' + \Big(\frac{2}{t^2} + \frac{1}{t}\Big)\, y = 0.$$

Abel's Theorem says that the Wronskian satisfies the equation

$$W'_{y_1 y_2}(t) - \Big(\frac{2}{t} + 1\Big)\, W_{y_1 y_2}(t) = 0.$$

This is a first order, linear equation for $W_{y_1 y_2}$. The integrating factor method implies

$$A(t) = -\int_{t_0}^{t} \Big(\frac{2}{s} + 1\Big)\, ds = -2\ln\Big(\frac{t}{t_0}\Big) - (t - t_0)$$

Second order variable coefficients, (2.1).

Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: $A(t) = -2\ln\Big(\dfrac{t}{t_0}\Big) - (t - t_0) = \ln\Big(\dfrac{t_0^2}{t^2}\Big) - (t - t_0).$

The integrating factor is $\mu = \dfrac{t_0^2}{t^2}\, e^{-(t-t_0)}$. Therefore,

$$\Big[\mu(t) W_{y_1 y_2}(t)\Big]' = 0 \quad \Rightarrow \quad \mu(t) W_{y_1 y_2}(t) - \mu(t_0) W_{y_1 y_2}(t_0) = 0$$

so, the solution is $W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, \dfrac{t^2}{t_0^2}\, e^{(t-t_0)}$.

Denoting $c = \big(W_{y_1 y_2}(t_0)/t_0^2\big)\, e^{-t_0}$, then $W_{y_1 y_2}(t) = c\, t^2 e^t$. $\triangleleft$