Mechanical and electrical oscillations (Sect. 2.5)

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Application: Mechanical Oscillations.
- Application: The RLC electrical circuit.

Remark:
Different physical systems may be mathematically identical.

Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Summary of solutions of the differential equation

$$y'' + a_1 y' + a_0 y = 0, \qquad a_1, a_2 \in \mathbb{R},$$

and characteristic roots $r_\pm = -\dfrac{a_1}{2} \pm \dfrac{1}{2}\sqrt{a_1^2 - 4a_0}$.

(1) Over damped systems: If $a_1^2 - 4a_0 > 0$, then,

$$y_1(t) = e^{r_+ t}, \qquad y_2(t) = e^{r_- t}.$$

(2) Critically damped systems: If $a_1^2 - 4a_0 = 0$, then,

$$y_1(t) = e^{-\frac{a_1}{2}\, t}, \qquad y_2(t) = t\, e^{-\frac{a_1}{2}\, t}.$$

(3) Under damped systems: If $a_1^2 - 4a_0 < 0$, then

$$y_1(t) = e^{\alpha t} \cos(\beta t), \qquad y_2(t) = e^{\alpha t} \sin(\beta t).$$

with $\alpha = -\dfrac{a_1}{2}$, $\beta = \dfrac{1}{2}\sqrt{4a_0 - a_1^2}$. Not damped: If $a_1 = 0$.

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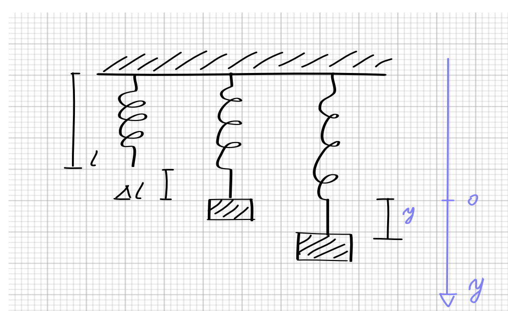
Application: Mechanical Oscillations.

Consider a spring attached to the ceiling, having rest length l, with an attached mass m.

- $(l + \Delta l)$ is called equilibrium position of the spring loaded with a mass m.
- The coordinate y measures vertical deviations from the equilibrium position.



Forces acting on the system:

- Weight: $F_g = mg$.
- Spring: $F_s = -k(\Delta l + y)$. Hooke's Law. (Small oscillations.)
- Damping: $F_d(t) = -d\, y'(t)$. Fluid Resistance.

Newton's Law: $m\, y''(t) = F_g + F_s(t) + F_d(t)$.

Application: Mechanical Oscillations.

Recall: $F_g = mg$, $F_s = -k(\Delta l + y)$, $F_d(t) = -d\, y'(t)$.

$$m\, y''(t) = F_g + F_s(t) + F_d(t).$$

That is, $$m\, y''(t) = mg - k(\Delta l + y(t)) - d\, y'(t).$$

At equilibrium, $y = 0$, $y' = 0$, then $k\, \Delta l = mg$. Hence

$$m\, y''(t) = -k\, y(t) - d\, y'(t)$$

$$m\, y'' + d\, y' + k\, y = 0.$$

To solve for the function y, we need the characteristic equation

$$m\, r^2 + d\, r + k = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2m}\big[-d \pm \sqrt{d^2 - 4mk}\big].$$

Application: Mechanical Oscillations.

Recall: $m\, y'' + d\, y' + k\, y = 0$, and $r_\pm = \frac{1}{2m}\big[-d \pm \sqrt{d^2 - 4mk}\big]$.

Not damped oscillations: $d = 0$. No fluid friction.

$$r_\pm = \pm\sqrt{-\frac{k}{m}}, \qquad \omega_0 = \sqrt{\frac{k}{m}}, \qquad r_\pm = \pm i\omega_0.$$

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

Remarks:

- Fundamental Frequency: ω_0; Period: $T = \frac{2\pi}{\omega_0}$.
- Equivalent expression: $y(t) = A \cos(\omega_0 t - \phi)$.
- Amplitude: A; Phase shift: ϕ.

Application: Mechanical Oscillations.

Recall: Not damped oscillations:

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \quad \Leftrightarrow \quad y(t) = A \cos(\omega_0 t - \phi).$$

where $\omega_0 = \sqrt{k/m}$ is the fundamental frequency, A is the amplitude, and ϕ the initial phase shift of the oscillations.
(Recall that the oscillation period is $T = \dfrac{2\pi}{\omega_0}$.)

Proof: Recall the trigonometric identity:

$$A \cos(\omega_0 t - \phi) = A \cos(\omega_0 t) \cos(\phi) + A \sin(\omega_0 t) \sin(\phi).$$

Therefore, comparing the first and last expressions above,

$$\left.\begin{array}{l} c_1 = A\cos(\phi) \\ c_2 = A\sin(\phi) \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} A = \sqrt{c_1^2 + c_2^2} \\ \phi = \arctan\Big(\dfrac{c_2}{c_1}\Big). \end{array}\right.$$

□

Application: Mechanical Oscillations.

Damped Oscillations

Recall: $m\,y'' + d\,y' + k\,y = 0$, and $r_{\pm} = \dfrac{1}{2m}\big[-d \pm \sqrt{d^2 - 4mk}\big]$.

Rewrite: $r_{\pm} = -\dfrac{d}{2m} \pm \sqrt{\Big(\dfrac{d}{2m}\Big)^2 - \dfrac{k}{m}}$.

Introduce: $\omega_0 = \sqrt{\dfrac{k}{m}}$, and $\omega_d = \dfrac{d}{2m}$. Hence

$$r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}.$$

Remark: We have three cases of damped oscillations:

(a) Over damped: $\omega_d > \omega_0$.

(b) Critically damped: $\omega_d = \omega_0$.

(c) Under damped: $\omega_d < \omega_0$.

Application: Mechanical Oscillations.

Recall: $m\,y'' + d\,y' + k\,y = 0$, and $r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}$.

(a) Over damped: $\omega_d > \omega_0$. Two distinct real roots:

$$y(t) = c_1 e^{r_+ t} + c_2\, e^{r_- t}.$$

(b) Critically damped: $\omega_d = \omega_0$. Repeated real root $r_+ = r_- = \hat{r}$:

$$y(t) = (c_1 + c_2 t)\, e^{\hat{r} t}.$$

(c) Under damped: $\omega_d < \omega_0$. Complex roots:

$$y(t) = \big[c_1 \cos(\beta t) + c_2 \sin(\beta t)\big]\, e^{-\omega_d t}$$

$$y(t) = A \cos(\beta t - \phi)\, e^{-\omega_d t}$$

where $r_{\pm} = -\omega_d \pm i\beta$, and $\beta = \sqrt{\omega_0^2 - \omega_d^2}$.

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg}/\text{Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg}/\text{Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: The equation is: $my'' + dy' + ky = 0$, with $m = 5$, $k = 5$, $d = 5$. The characteristic roots are

$$r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}, \quad \omega_d = \frac{d}{2m} = \frac{1}{2}, \quad \omega_0 = \sqrt{\frac{k}{m}} = 1.$$

$$r_{\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$ Under damped oscillations.

$$y(t) = A \cos\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big)\, e^{-t/2}.$$

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg}/\text{Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg}/\text{Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: Recall: $y(t) = A\cos\Big(\frac{\sqrt{3}}{2}\,t - \phi\Big)\,e^{-t/2}$. Hence,

$$y'(t) = -\frac{\sqrt{3}}{2}\,A\sin\Big(\frac{\sqrt{3}}{2}\,t - \phi\Big)\,e^{-t/2} - \frac{1}{2}\,A\cos\Big(\frac{\sqrt{3}}{2}\,t - \phi\Big)\,e^{-t/2}.$$

The initial conditions:

$$\sqrt{3} = y(0) = A\cos(\phi), \quad 0 = y'(0) = \frac{\sqrt{3}}{2}\,A\sin(\phi) - \frac{1}{2}\,A\cos(\phi).$$

$$\tan(\phi) = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \phi = \frac{\pi}{6}, \quad \Rightarrow \quad A = 2.$$

We conclude: $y(t) = 2\cos\Big(\frac{\sqrt{3}}{2}\,t - \frac{\pi}{6}\Big)\,e^{-t/2}$. $\triangleleft$

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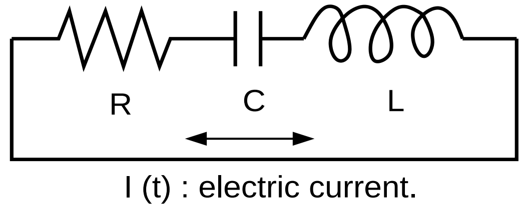
The RLC electrical circuit.

Consider an electric circuit with resistance R, non-zero capacitor C, and non-zero inductance L, as in the figure.



Kirchhoff's Law: The electric current flowing in the circuit satisfies:

$$L\,I'(t) + R\,I(t) + \frac{1}{C}\int_{t_0}^{t} I(s)\,ds = 0.$$

Derivate both sides above: $L\,I''(t) + R\,I'(t) + \frac{1}{C}\,I(t) = 0.$

Divide by L: $I''(t) + 2\Big(\frac{R}{2L}\Big)\,I'(t) + \frac{1}{LC}\,I(t) = 0.$

Introduce $\alpha = \frac{R}{2L}$ and $\omega = \frac{1}{\sqrt{LC}}$, then $I'' + 2\alpha\,I' + \omega^2\,I = 0.$

The RLC electrical circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\,I' + \omega^2\,I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2}\big[-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}\big] \quad\Rightarrow\quad r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$

Case (a) $R = 0$. This implies $\alpha = 0$, so $r_\pm = \pm i\omega$. Therefore,

$$I_1(t) = \cos(\omega t), \qquad I_2(t) = \sin(\omega t).$$

Remark: When the circuit has no resistance, the current oscillates without dissipation.

The RLC electrical circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

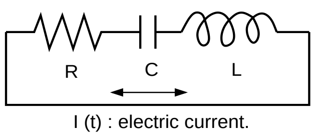
Solution: Recall: $r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

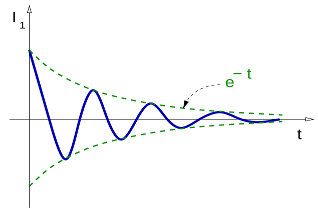
Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \quad \Leftrightarrow \quad \frac{R^2}{4L^2} < \frac{1}{LC} \quad \Leftrightarrow \quad \alpha^2 < \omega^2.$$

Therefore, $r_\pm = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos\big(\sqrt{\omega^2 - \alpha^2}\, t\big), \quad I_2(t) = e^{-\alpha t} \sin\big(\sqrt{\omega^2 - \alpha^2}\, t\big).$$





The resistance R damps the current oscillations.