Second order linear ODE (Sect. 2.3).

- Review: Second order linear differential equations.
- Idea: Soving constant coefficients equations.
- The characteristic equation.
- Main result for constant coefficients equations.
- Characteristic polynomial with complex roots.

Review: Second order linear ODE.

Definition

(a) A *second order linear differential equation* in the unknown y is

$$L(y) = y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad (1)$$

(b) Eq. (1) is called *homogeneous* iff the $b = 0$.

(c) Eq. (1) is called of *constant coefficients* iff a_1 and a_0 are constants; otherwise is called of *variable coefficients*.

(d) The functions y_1 and y_2 are *fundamental solutions* of $L(y) = 0$ iff $L(y_1) = 0$, $L(y_2) = 0$ and y_1, y_2 are linearly independent.

(e) The *general solution* of the homogeneous equation $L(y) = 0$ denotes any function $y_{\rm gen}$ that can be written as

$$y_{\rm gen}(t) = c_1\, y_1(t) + c_2\, y_2(t),$$

where y_1, y_2 are fundamental solutions of $L(y) = 0$ and c_1, c_2, are arbitrary constants.

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Idea: Soving constant coefficients equations.

Remark: Just by trial and error one can find solutions to second order, constant coefficients, homogeneous, linear differential equations. We present the main ideas with an example.

Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: We look for solutions proportional to exponentials e^{rt}, for an appropriate constant $r \in \mathbb{R}$, since the exponential can be canceled out from the equation.
If $y(t) = e^{rt}$, then $y'(t) = re^{rt}$, and $y''(t) = r^2 e^{rt}$. Hence

$$(r^2 + 5r + 6)e^{rt} = 0 \quad \Leftrightarrow \quad r^2 + 5r + 6 = 0.$$

That is, r must be a root of the polynomial $p(r) = r^2 + 5r + 6$.

This polynomial is called the characteristic polynomial of the differential equation.

Idea: Soving constant coefficients equations.

Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

The roots of the characteristic polynomial are

$$r = \frac{1}{2}\left(-5 \pm \sqrt{25 - 24}\right) = \frac{1}{2}\left(-5 \pm 1\right) \quad \Rightarrow \quad \begin{cases} r_1 = -2, \\ r_2 = -3. \end{cases}$$

Therefore, we have found the fundamental solutions

$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{-3t}.$$

Therefore, the general solution is

$$y_{\text{gen}}(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}. \qquad \lhd$$

Idea: Soving constant coefficients equations.

Summary: The differential equation $y'' + 5y' + 6y = 0$ has infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Remarks:

- There are two free constants in the solution found above.
- The ODE above is second order, so two integrations must be done to find the solution. This explain the origin of the two free constant in the solution.
- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.

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The characteristic equation.

Definition

Given a second order linear homogeneous differential equation with constant coefficients

$$y'' + a_1 y' + a_0 = 0, \qquad (2)$$

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (2) are, respectively,

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

Remark: If $r_1 \neq r_2$ are the solutions of the characteristic equation and c_1, c_2 are constants, then the general solution of Eq. (2) is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

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Main result for constant coefficients equations.

Theorem (Constant coefficients)

Given real constants a_1, a_0, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1\, y' + a_0\, y = 0.$$

Let r_+, r_- be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let c_0, c_1 be arbitrary constants. Then, the general solution of the differential eqation is given by:

(a) *If $r_+ \neq r_-$, real or complex, then $y(t) = c_0\, e^{r_+ t} + c_1\, e^{r_- t}$.*

(b) *If $r_+ = r_- = \hat{r} \in \mathbb{R}$, then is $y(t) = c_0\, e^{\hat{r} t} + c_1\, t e^{\hat{r} t}$.*

Furthermore, given real constants t_0, y_0 and y_1, there is a unique solution to the initial value problem

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

Main result for constant coefficients equations.

Example

Find the solution y of the initial value problem

$$y'' + 5y' + 6 = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

Solution: A solution of the differential equation above is

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

We now find the constants c_1 and c_2 that satisfy the initial conditions above:

$$1 = y(0) = c_1 + c_2, \qquad -1 = y'(0) = -2c_1 - 3c_2.$$

$$c_1 = 1 - c_2 \Rightarrow 1 = 2(1 - c_2) + 3c_2 \Rightarrow c_2 = -1 \Rightarrow c_1 = 2.$$

Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

$\triangleleft$

Main result for constant coefficients equations.

Example

Find the general solution y of the differential equation

$$2y'' - 3y' + y = 0.$$

Solution: We look for every solution of the form $y(t) = e^{rt}$, where r is a solution of the characteristic equation

$$2r^2 - 3r + 1 = 0 \;\Rightarrow\; r = \frac{1}{4}\big(3 \pm \sqrt{9-8}\big) \;\Rightarrow\; \begin{cases} r_1 = 1, \\ r_2 = \dfrac{1}{2}. \end{cases}$$

Therefore, the general solution of the equation above is

$$y(t) = c_1 e^{t} + c_2 e^{t/2},$$

where c_1, c_2 are arbitrary constants.

$\triangleleft$

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Characteristic polynomial with complex roots.

Example

Find the general solution of the equation $y'' - 2y' + 6y = 0$.

Solution: We first find the roots of the characteristic polynomial,

$$r^2 - 2r + 6 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(2 \pm \sqrt{4 - 24}\big) \quad \Rightarrow \quad r_\pm = 1 \pm i\sqrt{5}.$$

A fundamental solution set is

$$\tilde{y}_1(t) = e^{(1+i\sqrt{5})\,t}, \qquad \tilde{y}_2(t) = e^{(1-i\sqrt{5})\,t}.$$

These are complex-valued functions. The general solution is

$$y(t) = \tilde{c}_1\, e^{(1+i\sqrt{5})\,t} + \tilde{c}_2\, e^{(1-i\sqrt{5})\,t}, \qquad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}. \qquad \triangleleft$$

Characteristic polynomial with complex roots.

- Complex numbers have the form $z = a + ib$, where $i^2 = -1$.
- The complex conjugate of z is the number $\overline{z} = a - ib$.
- $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$ are the real and imaginary parts of z
- Hence: $\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$
- $e^{a+ib} = \displaystyle\sum_{n=0}^{\infty} \frac{(a+ib)^n}{n!}$. In particular holds $e^{a+ib} = e^a\, e^{ib}$.
- Euler's formula: $e^{ib} = \cos(b) + i\sin(b)$.
- Hence, a complex number of the form e^{a+ib} can be written as

$$e^{a+ib} = e^a\big[\cos(b) + i\sin(b)\big], \quad e^{a-ib} = e^a\big[\cos(b) - i\sin(b)\big].$$

- From e^{a+ib} and e^{a-ib} we get the real numbers

$$\frac{1}{2}\big(e^{a+ib} + e^{a-ib}\big) = e^a \cos(b), \quad \frac{1}{2i}\big(e^{a+ib} - e^{a-ib}\big) = e^a \sin(b).$$

Characteristic polynomial with complex roots.

Remark:

- The solutions found above include real-valued and complex-valued solutions.
- Since the differential equation is real-valued, it is usually important in applications to obtain the most general real-valued solution. (See RLC circuit in Applications.)
- In the expression above it is difficult to take apart real-valued solutions from complex-valued solutions.
- In other words: It is not simple to see what values of $\tilde{c}_1$ and $\tilde{c}_2$ make the general solution above to be real-valued.
- One way to find the real-valued general solution is to find real-valued fundamental solutions.

Characteristic polynomial with complex roots.

Theorem (Complex roots)

If the constants $a_1, a_0 \in \mathbb{R}$ satisfy that $a_1^2 - 4a_0 < 0$, then the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$ of the equation

$$y'' + a_1\, y' + a_0\, y = 0 \tag{3}$$

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{a_1}{2}, \qquad \beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}.$$

Furthermore, a fundamental set of solutions to Eq. (3) is

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$

while another fundamental set of solutions to Eq. (3) is

$$y_1(t) = e^{\alpha t}\cos(\beta t), \qquad y_2(t) = e^{\alpha t}\sin(\beta t).$$

Characteristic polynomial with complex roots.

Idea of the Proof: Recall that the functions

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$

are solutions to $y'' + a_1\, y' + a_0\, y = 0$. Also recall that

$$\tilde{y}_1(t) = e^{\alpha t}\big[\cos(\beta t) + i\sin(\beta t)\big], \quad \tilde{y}_2(t) = e^{\alpha t}\big[\cos(\beta t) - i\sin(\beta t)\big].$$

Then the functions

$$y_1(t) = \frac{1}{2}\big(\tilde{y}_1(t) + \tilde{y}_2(t)\big) \qquad y_2(t) = \frac{1}{2i}\big(\tilde{y}_1(t) - \tilde{y}_2(t)\big)$$

are also solutions to the same differential equation. We conclude that y_1 and y_2 are real valued and

$$y_1(t) = e^{\alpha t}\cos(\beta t), \qquad y_2(t) = e^{\alpha t}\sin(\beta t).$$

□

Characteristic polynomial with complex roots.

Example

Find the real-valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$

Solution: Recall: Complex valued solutions are

$$\tilde{y}_1(t) = e^{(1+i\sqrt{5})\,t}, \qquad \tilde{y}_2(t) = e^{(1-i\sqrt{5})\,t}.$$

Any linear combination of these functions is solution of the differential equation. In particular,

$$y_1(t) = \frac{1}{2}\big[\tilde{y}_1(t) + \tilde{y}_2(t)\big], \quad y_2(t) = \frac{1}{2i}\big[\tilde{y}_1(t) - \tilde{y}_2(t)\big].$$

Now, recalling $e^{(1\pm i\sqrt{5})\,t} = e^t e^{\pm i\sqrt{5}\,t}$

$$y_1(t) = \frac{1}{2}\big[e^t\, e^{i\sqrt{5}t} + e^t\, e^{-i\sqrt{5}t}\big], \quad y_2(t) = \frac{1}{2i}\big[e^t\, e^{i\sqrt{5}t} - e^t\, e^{-i\sqrt{5}t}\big],$$

Characteristic polynomial with complex roots.

Example

Find the real-valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$

Solution: $y_1 = \dfrac{e^t}{2}\big[e^{i\sqrt{5}t} + e^{-i\sqrt{5}t}\big]$, $y_2 = \dfrac{e^t}{2i}\big[e^{i\sqrt{5}t} - e^{-i\sqrt{5}t}\big]$.

The Euler formula and its complex-conjugate formula

$$e^{i\sqrt{5}t} = \big[\cos(\sqrt{5}\,t) + i\sin(\sqrt{5}\,t)\big],$$

$$e^{-i\sqrt{5}\,t} = \big[\cos(\sqrt{5}\,t) - i\sin(\sqrt{5}\,t)\big],$$

imply the inverse relations

$$e^{i\sqrt{5}t} + e^{-i\sqrt{5}t} = 2\cos(\sqrt{5}t), \quad e^{i\sqrt{5}t} - e^{-i\sqrt{5}t} = 2i\sin(\sqrt{5}t).$$

So functions y_1 and y_2 can be written as

$$y_1(t) = e^t\cos(\sqrt{5}\,t), \qquad y_2(t) = e^t\sin(\sqrt{5}\,t).$$

Characteristic polynomial with complex roots.

Example

Find the real-valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$

Solution: Recall: $y(t) = \tilde{c}_1 e^{(1+i\sqrt{5})\,t} + \tilde{c}_2 e^{(1-i\sqrt{5})\,t}$, $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}$.

The calculation above says that a real-valued fundamental set is

$$y_1(t) = e^t \cos(\sqrt{5}\,t), \qquad y_2(t) = e^t \sin(\sqrt{5}\,t).$$

Hence, the complex-valued general solution can also be written as

$$y(t) = \big[c_1 \cos(\sqrt{5}\,t) + c_2 \sin(\sqrt{5}\,t)\big]\, e^t, \qquad c_1, c_2 \in \mathbb{C}.$$

The real-valued general solution is simple to obtain:

$$y(t) = \big[c_1 \cos(\sqrt{5}\,t) + c_2 \sin(\sqrt{5}\,t)\big]\, e^t, \qquad c_1, c_2 \in \mathbb{R}.$$

We just restricted the coefficients c_1, c_2 to be real-valued. $\triangleleft$

Characteristic polynomial with complex roots.

Example

Show that $y_1(t) = e^t \cos(\sqrt{5}\,t)$ and $y_2(t) = e^t \sin(\sqrt{5}\,t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

Solution: $y_1(t) = e^t \cos(\sqrt{5}\,t)$, $y_2(t) = e^t \sin(\sqrt{5}\,t)$.

Summary:

- These functions are solutions of the differential equation.
- They are not proportional to each other, Hence li.
- Therefore, y_1, y_2 form a fundamental set.
- The general solution of the equation is

$$y(t) = \big[c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)\big]\, e^t.$$

- y is real-valued for c_1, $c_2 \in \mathbb{R}$.
- y is complex-valued for c_1, $c_2 \in \mathbb{C}$.

Characteristic polynomial with complex roots.

Example

Find real-valued fundamental solutions to the equation

$$y'' + 2\,y' + 6\,y = 0.$$

Solution:

The roots of the characteristic polynomial $p(r) = r^2 + 2r + 6$ are

$$r_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4-24}\big] = \frac{1}{2}\big[-2 \pm \sqrt{-20}\big] \;\Rightarrow\; r_\pm = -1 \pm i\sqrt{5}.$$

These are complex-valued roots, with

$$\alpha = -1, \qquad \beta = \sqrt{5}.$$

Real-valued fundamental solutions are

$$y_1(t) = e^{-t}\cos(\sqrt{5}\,t), \qquad y_2(t) = e^{-t}\sin(\sqrt{5}\,t). \qquad \triangleleft$$

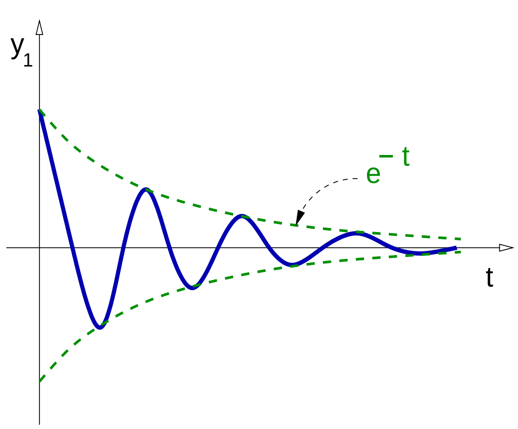
Characteristic polynomial with complex roots.

Example

Find real-valued fundamental solutions to the equation

$$y'' + 2\,y' + 6\,y = 0.$$

Solution: $y_1(t) = e^{-t}\cos(\sqrt{5}\,t)$, $y_2(t) = e^{-t}\sin(\sqrt{5}\,t)$.



Differential equations like the one in this example describe physical processes related to damped oscillations. For example pendulums with friction.

Characteristic polynomial with complex roots.

Example

Find the real-valued general solution of $y'' + 5\,y = 0$.

Solution: The characteristic polynomial is $p(r) = r^2 + 5$.

Its roots are $r_{\pm} = \pm\sqrt{5}\,i$. This is the case $\alpha = 0$, and $\beta = \sqrt{5}$.

Real-valued fundamental solutions are

$$y_1(t) = \cos(\sqrt{5}\,t), \qquad y_2(t) = \sin(\sqrt{5}\,t).$$

The real-valued general solution is

$$y(t) = c_1 \cos(\sqrt{5}\,t) + c_2 \sin(\sqrt{5}\,t), \qquad c_1,\, c_2 \in \mathbb{R}. \qquad \triangleleft$$

Remark: Equations like the one in this example describe oscillatory physical processes without dissipation, $\alpha = 0$.