## On linear and non-linear equations. (Sect. 1.6).

- Review: Linear differential equations.
- Non-linear differential equations.
- The Picard-Lindelöf Theorem.
- Properties of solutions to non-linear ODE.
- Direction Fields.

## Review: Linear differential equations.

### Theorem (Variable coefficients)

*Given continuous functions $a, b : (t_1, t_2) \to \mathbb{R}$, with $t_2 > t_1$, and given constants $t_0 \in (t_1, t_2)$, $y_0 \in \mathbb{R}$, the IVP*

$$y' = -a(t)\, y + b(t), \qquad y(t_0) = y_0,$$

*has the unique solution $y : (t_1, t_2) \to \mathbb{R}$ given by*

$$y(t) = \frac{1}{\mu(t)} \Big[ y_0 + \int_{t_0}^{t} \mu(s)\, b(s)\, ds \Big], \tag{1}$$

*where the integrating factor function is given by*

$$\mu(t) = e^{A(t)}, \qquad A(t) = \int_{t_0}^{t} a(s)\, ds.$$

Proof: Based on the integration factor method.

## Review: Linear differential equations.

Remarks:

- The Theorem above assumes that the coefficients $a$, $b$, are continuous in $(t_1, t_2) \subset \mathbb{R}$.
- The Theorem above implies:
  - (a) There is an explicit expression for the solutions of a linear IVP, given in Eq. (1).
  - (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  - (c) For every initial condition $y_0 \in \mathbb{R}$ the corresponding solution $y(t)$ of a linear IVP is defined for all $t \in (t_1, t_2)$.
- None of these properties holds for solutions to non-linear differential equations.

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# Non-linear differential equations.

## Definition

An ordinary differential equation $y'(t) = f(t, y(t))$ is called *non-linear* iff the function $f$ is non-linear in the second argument.

## Example

(a) The differential equation $y'(t) = \dfrac{t^2}{y^3(t)}$ is non-linear, since the function $f(t,u) = t^2/u^3$ is non-linear in the second argument.

(b) The differential equation $y'(t) = 2ty(t) + \ln\big(y(t)\big)$ is non-linear, since the function $f(t,u) = 2tu + \ln(u)$ is non-linear in the second argument, due to the term $\ln(u)$.

(c) The differential equation $\dfrac{y'(t)}{y(t)} = 2t^2$ is linear, since the function $f(t,u) = 2t^2 u$ is linear in the second argument.

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## The Picard-Lindelöf Theorem.

**Theorem (Picard-Lindelöf)**

*Consider the initial value problem*

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

*If* $f : S \to \mathbb{R}$ *is continuous on the square*

$$S = [t_0 - a, t_0 + a] \times [y_0 - a, y_0 + a] \subset \mathbb{R}^2,$$

*for some* $a > 0$*, and satisfies the Lipschitz condition that there exists* $k > 0$ *such that*

$$|f(t, y_2) - f(t, y_1)| < k\,|y_2 - y_1|,$$

*for all* $(t, y_2)$*,* $(t, y_1) \in S$*, then there exists a positive* $b < a$ *such that there exists a unique solution* $y : [t_0 - b, t_0 + b] \to \mathbb{R}$ *to the IVP above.*

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## Properties of solutions to non-linear ODE.

Recall: The non-linear initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

has a unique solution in a region small enough near the initial data.

Remarks:

(i) There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.

(ii) Non-uniqueness of solution to the IVP above may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.

(iii) Changing the initial data $y_0$ may change the domain on the variable $t$ where the solution $y(t)$ is defined.

## Properties of solutions to non-linear ODE.

Example

Given non-zero constants $a_1$, $a_2$, $a_3$, $a_4$, find every solution $y$ of

$$y' = \frac{t^2}{\left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right)}.$$

Solution: The ODE is separable. So first, rewrite the equation as

$$\left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right) y' = t^2,$$

then we integrate in $t$ on both sides of the equation,

$$\int \left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right) y'\, dt = \int t^2\, dt + c.$$

Introduce the substitution $u = y(t)$, so $du = y'(t)\, dt$,

$$\int \left(u^4 + a_4\, u^3 + a_3\, u^2 + a_2\, u + a_1\right) du = \int t^2\, dt + c.$$

## Properties of solutions to non-linear ODE.

Example

Given non-zero constants $a_1$, $a_2$, $a_3$, $a_4$, find every solution $y$ of

$$y' = \frac{t^2}{\big(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\big)}.$$

Solution:

Recall: $\displaystyle\int \big(u^4 + a_4\, u^3 + a_3\, u^2 + a_2\, u + a_1\big)\, du = \int t^2\, dt + c.$

Integrate, and in the result substitute back the function $y$:

$$\frac{1}{5}\, y^5(t) + \frac{a_4}{4}\, y^4(t) + \frac{a_3}{3}\, y^3(t) + \frac{a_2}{2}\, y^2(t) + a_1\, y(t) = \frac{t^3}{3} + c.$$

The solution is in implicit form. It is the root of a polynomial degree five. There is no formula for the roots of a general polynomial degree five or bigger.

There is no explicit expression for solutions $y$ of the ODE. $\triangleleft$

## Properties of solutions to non-linear ODE.

Example

Find every solution $y$ of the initial value problem

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

Remark: The equation above is non-linear, separable, and the function $f(t,u) = u^{1/3}$ has derivative

$$\partial_u f = \frac{1}{3}\, \frac{1}{u^{2/3}},$$

so $\partial_u f$ is not continuous at $u = 0$.

The initial condition above is precisely where $f$ is not continuous.

Solution: There are two solutions to the IVP above:

The first solution is

$$y_1(t) = 0.$$

## Properties of solutions to non-linear ODE.

Example

Find every solution $y$ of the initial value problem

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

Solution: The second solution is obtained as follows:

$$\int \big[y(t)\big]^{-1/3}\, y'(t)\, dt = \int dt + c.$$

Then, the substitution $u = y(t)$, with $du = y'(t)\, dt$, implies that

$$\int u^{-1/3}\, du = \int dt + c \quad \Rightarrow \quad \frac{3}{2}\big[y(t)\big]^{2/3} = t + c,$$

$$y(t) = \Big[\frac{2}{3}(t + c)\Big]^{3/2} \Rightarrow 0 = y(0) = \Big(\frac{2}{3}c\Big)^{3/2} \Rightarrow c = 0.$$

So, the second solution is: $y_2(t) = \Big(\frac{2}{3}t\Big)^{3/2}$. Recall $y_1(t) = 0$. $\triangleleft$

## Properties of solutions to non-linear ODE.

Example

Find the solution $y$ to the initial value problem

$$y'(t) = y^2(t), \qquad y(0) = y_0.$$

Solution: This is a separable equation. So,

$$\int \frac{y'\, dt}{y^2} = \int dt + c \quad \Rightarrow \quad -\frac{1}{y} = t + c \quad \Rightarrow \quad y(t) = -\frac{1}{t + c}.$$

Using the initial condition in the expression above,

$$y_0 = y(0) = -\frac{1}{c} \quad \Rightarrow \quad c = -\frac{1}{y_0} \quad \Rightarrow \quad y(t) = \frac{1}{\Big(\frac{1}{y_0} - t\Big)}.$$

This solution diverges at $t = 1/y_0$, so its domain is $\mathbb{R} - \{y_0\}$.

The solution domain depends on the values of the initial data $y_0$. $\triangleleft$

## Properties of solutions to non-linear ODE.

### Summary:

- Linear ODE:
  (a) There is an explicit expression for the solution of a linear IVP.
  (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  (c) The domain of the solution of a linear IVP is defined for every initial condition $y_0 \in \mathbb{R}$.

- Non-linear ODE:
  (i) There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.
  (ii) Non-uniqueness of solution to a non-linear IVP may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.
  (iii) Changing the initial data $y_0$ may change the domain on the variable $t$ where the solution $y(t)$ is defined.

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- **Direction Fields.**

## Direction Fields.

### Remarks:

- One does not need to solve a differential equation $y'(t) = f(t, y(t))$ to have a qualitative idea of the solution.
- Recall that $y'(t)$ represents the slope of the tangent line to the graph of function $y$ at the point $(t, y(t))$.
- A differential equation provides these slopes, $f(t, y(t))$, for every point $(t, y(t))$.
- Key idea: Graph the function $f(t, y)$ on the $yt$-plane, not as points, but as slopes of small segments.

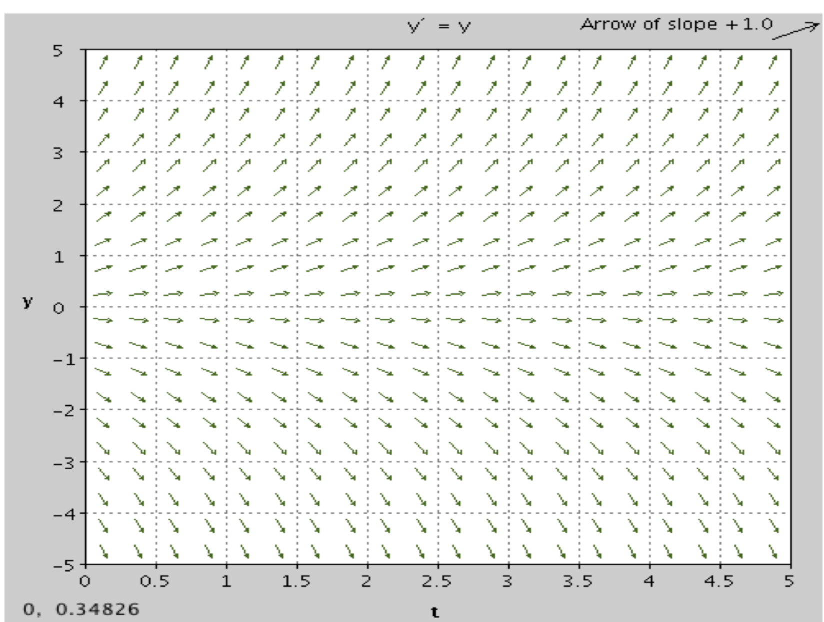
### Definition

A *Direction Field* for the differential equation $y'(t) = f(t, y(t))$ is the graph on the $yt$-pane of the values $f(t, y)$ as slopes of a small segments.

## Direction Fields.

### Example

We know that the solution of $y' = y$ are the exponentials $y(t) = y_0\, e^t$. The graph of these solution is simple.
So is the direction field:



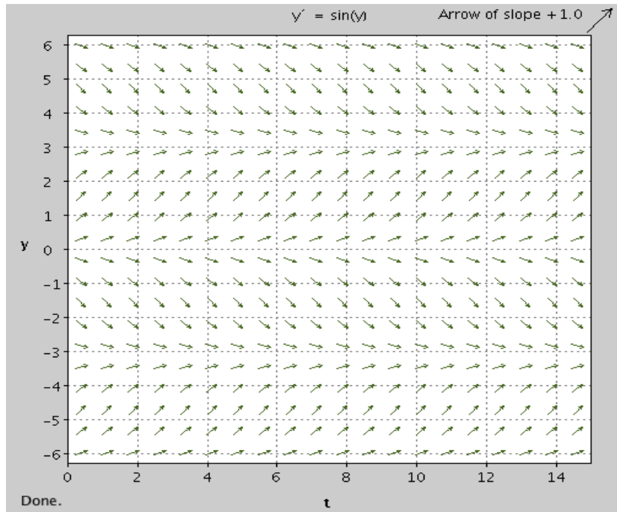
## Direction Fields.

### Example

The solution of $y' = \sin(y)$ is simple to compute. The equation is separable. After some calculations the implicit solution are

$$\ln\left|\frac{\csc(y_0)+\cot(y)}{\csc(y)+\cot(y)}\right| = t.$$

for $y_0 \in \mathbb{R}$. The graph of these solution is not simple to do.
But the direction field is simple to plot:




## Direction Fields.

### Example

The solution of $y' = \dfrac{(1+y^3)}{(1+t^2)}$ could be hard to compute. But the direction field is simple to plot: