On linear and non-linear equations. (Sect. 1.6).

- Review: Linear differential equations.
- Non-linear differential equations.
- The Picard-Lindelöf Theorem.
- Properties of solutions to non-linear ODE.
- The Proof of Picard-Lindelöf's Theorem.
- Direction Fields.

Review: Linear differential equations.

Theorem (Variable coefficients)

Given continuous functions $a, b : (t_1, t_2) \to \mathbb{R}$, *with* $t_2 > t_1$, *and given constants* $t_0 \in (t_1, t_2)$, $y_0 \in \mathbb{R}$, *the IVP*

$$y' = -a(t)\, y + b(t), \qquad y(t_0) = y_0,$$

has the unique solution $y : (t_1, t_2) \to \mathbb{R}$ *given by*

$$y(t) = \frac{1}{\mu(t)} \Big[y_0 + \int_{t_0}^{t} \mu(s)\, b(s)\, ds \Big], \tag{1}$$

where the integrating factor function is given by

$$\mu(t) = e^{A(t)}, \qquad A(t) = \int_{t_0}^{t} a(s)\, ds.$$

Proof: Based on the integration factor method.

Review: Linear differential equations.

Remarks:

- The Theorem above assumes that the coefficients a, b, are continuous in $(t_1, t_2) \subset \mathbb{R}$.
- The Theorem above implies:
  - (a) There is an explicit expression for the solutions of a linear IVP, given in Eq. (1).
  - (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  - (c) For every initial condition $y_0 \in \mathbb{R}$ the corresponding solution $y(t)$ of a linear IVP is defined for all $t \in (t_1, t_2)$.
- None of these properties holds for solutions to non-linear differential equations.

On linear and non-linear equations. (Sect. 1.6).

- Review: Linear differential equations.
- **Non-linear differential equations.**
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Non-linear differential equations.

Definition

An ordinary differential equation $y'(t) = f(t, y(t))$ is called *non-linear* iff the function f is non-linear in the second argument.

Example

(a) The differential equation $y'(t) = \dfrac{t^2}{y^3(t)}$ is non-linear, since the function $f(t,u) = t^2/u^3$ is non-linear in the second argument.

(b) The differential equation $y'(t) = 2ty(t) + \ln\big(y(t)\big)$ is non-linear, since the function $f(t,u) = 2tu + \ln(u)$ is non-linear in the second argument, due to the term $\ln(u)$.

(c) The differential equation $\dfrac{y'(t)}{y(t)} = 2t^2$ is linear, since the function $f(t,u) = 2t^2 u$ is linear in the second argument.

On linear and non-linear equations. (Sect. 1.6).

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The Picard-Lindelöf Theorem.

Theorem (Picard-Lindelöf)

Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

If $f : S \to \mathbb{R}$ is continuous on the square

$$S = [t_0 - a, t_0 + a] \times [y_0 - a, y_0 + a] \subset \mathbb{R}^2,$$

for some $a > 0$, and satisfies the Lipschitz condition that there exists $k > 0$ such that

$$|f(t, y_2) - f(t, y_1)| < k\,|y_2 - y_1|,$$

for all (t, y_2), $(t, y_1) \in S$, then there exists a positive $b < a$ such that there exists a unique solution $y : [t_0 - b, t_0 + b] \to \mathbb{R}$ to the IVP above.

On linear and non-linear equations. (Sect. 1.6).

- Review: Linear differential equations.
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Properties of solutions to non-linear ODE.

Recall: The non-linear initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

has a unique solution in a region small enough near the initial data.

Remarks:

(i) There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.

(ii) Non-uniqueness of solution to the IVP above may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.

(iii) Changing the initial data y_0 may change the domain on the variable t where the solution $y(t)$ is defined.

Properties of solutions to non-linear ODE.

Example

Given non-zero constants a_1, a_2, a_3, a_4, find every solution y of

$$y' = \frac{t^2}{\left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right)}.$$

Solution: The ODE is separable. So first, rewrite the equation as

$$\left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right) y' = t^2,$$

then we integrate in t on both sides of the equation,

$$\int \left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right) y'\, dt = \int t^2\, dt + c.$$

Introduce the substitution $u = y(t)$, so $du = y'(t)\, dt$,

$$\int \left(u^4 + a_4\, u^3 + a_3\, u^2 + a_2\, u + a_1\right) du = \int t^2\, dt + c.$$

Properties of solutions to non-linear ODE.

Example

Given non-zero constants a_1, a_2, a_3, a_4, find every solution y of

$$y' = \frac{t^2}{\left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right)}.$$

Solution:

Recall: $\displaystyle\int \left(u^4 + a_4\, u^3 + a_3\, u^2 + a_2\, u + a_1\right) du = \int t^2\, dt + c.$

Integrate, and in the result substitute back the function y:

$$\frac{1}{5}\, y^5(t) + \frac{a_4}{4}\, y^4(t) + \frac{a_3}{3}\, y^3(t) + \frac{a_2}{2}\, y^2(t) + a_1\, y(t) = \frac{t^3}{3} + c.$$

The solution is in implicit form. It is the root of a polynomial degree five. There is no formula for the roots of a general polynomial degree five or bigger.

There is no explicit expression for solutions y of the ODE. $\lhd$

Properties of solutions to non-linear ODE.

Example

Find every solution y of the initial value problem

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

Remark: The equation above is non-linear, separable, and the function $f(t, u) = u^{1/3}$ has derivative

$$\partial_u f = \frac{1}{3}\, \frac{1}{u^{2/3}},$$

so $\partial_u f$ is not continuous at $u = 0$.

The initial condition above is precisely where f is not continuous.

Solution: There are two solutions to the IVP above:

The first solution is

$$y_1(t) = 0.$$

Properties of solutions to non-linear ODE.

Example

Find every solution y of the initial value problem

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

Solution: The second solution is obtained as follows:

$$\int \big[y(t)\big]^{-1/3} y'(t)\, dt = \int dt + c.$$

Then, the substitution $u = y(t)$, with $du = y'(t)\, dt$, implies that

$$\int u^{-1/3}\, du = \int dt + c \quad \Rightarrow \quad \frac{3}{2}\big[y(t)\big]^{2/3} = t + c,$$

$$y(t) = \Big[\frac{2}{3}(t + c)\Big]^{3/2} \Rightarrow 0 = y(0) = \Big(\frac{2}{3}c\Big)^{3/2} \Rightarrow c = 0.$$

So, the second solution is: $y_2(t) = \Big(\frac{2}{3}t\Big)^{3/2}$. Recall $y_1(t) = 0$. $\lhd$

Properties of solutions to non-linear ODE.

Example

Find the solution y to the initial value problem

$$y'(t) = y^2(t), \qquad y(0) = y_0.$$

Solution: This is a separable equation. So,

$$\int \frac{y'\, dt}{y^2} = \int dt + c \quad \Rightarrow \quad -\frac{1}{y} = t + c \quad \Rightarrow \quad y(t) = -\frac{1}{t + c}.$$

Using the initial condition in the expression above,

$$y_0 = y(0) = -\frac{1}{c} \quad \Rightarrow \quad c = -\frac{1}{y_0} \quad \Rightarrow \quad y(t) = \frac{1}{\Big(\frac{1}{y_0} - t\Big)}.$$

This solution diverges at $t = 1/y_0$, so its domain is $\mathbb{R} - \{y_0\}$.

The solution domain depends on the values of the initial data y_0.$\lhd$

Properties of solutions to non-linear ODE.

Summary:

- Linear ODE:
  (a) There is an explicit expression for the solution of a linear IVP.
  (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  (c) The domain of the solution of a linear IVP is defined for every initial condition $y_0 \in \mathbb{R}$.
- Non-linear ODE:
  (i) There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.
  (ii) Non-uniqueness of solution to a non-linear IVP may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.
  (iii) Changing the initial data y_0 may change the domain on the variable t where the solution $y(t)$ is defined.

On linear and non-linear equations. (Sect. 1.6).

- Review: Linear differential equations.
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- **The Proof of Picard-Lindelöf's Theorem.**
- Direction Fields.

The Proof of Picard-Lindelöf's Theorem.

Remark: Idea of the proof of Picard-Lindelöf's Theorem.

(a) Transform the differential equation into an integral equation.

$$y'(t) = f(t, y(t)), \quad y(0) = y_0 \quad \longrightarrow \quad \text{Integral equation.}$$

(b) Use the integral equation to define a sequence $\{y_n(t)\}_{n=0}^{\infty}$ of approximate solutions.

(c) Show that the sequence of approximate solutions converges to the solution of the equation.

$$\lim_{n\to\infty} y_n(t) = y(t)$$

(d) The main technique used in the convergence statement is the Contraction Fixed Point Theorem in Banach Spaces.

The Proof of Picard-Lindelöf's Theorem.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = t\,y, \qquad y(0) = 1.$$

Solution: First notice that the equation is separable. So it is simple to find the solution following Section 1.3,

$$\frac{y'}{y} = t \quad \Rightarrow \quad \ln(y) = \frac{t^2}{2} + c \quad \Rightarrow \quad y(t) = \tilde{c}\, e^{t^2/2}.$$

The initial condition implies,

$$1 = y(0) = \tilde{c} \quad \Rightarrow \quad y(t) = e^{t^2/2}.$$

In the next slide we use Picard-Lindelöf's idea.

The Proof of Picard-Lindelöf's Theorem.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = t\,y, \qquad y(0) = 1.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t s\,y(s)\,ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t s\,y(s)\,ds.$$

Using the initial condition, $y(0) = 1$,

$$y(t) = 1 + \int_0^t s\,y(s)\,ds.$$

This is the integral equation.

The Proof of Picard-Lindelöf's Theorem.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = t\,y, \qquad y(0) = 1.$$

Solution: Integral equation: $y(t) = 1 + \int_0^t s\,y(s)\,ds$.

We now define the sequence of approximate solutions:

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t s\,y_n(s)\,ds, \quad n \geqslant 0.$$

We now compute the first elements in the sequence.

$$n = 0, \quad y_1(t) = 1 + \int_0^t s\,y_0(s)\,ds = 1 + \int_0^t s\,ds = 1 + \frac{t^2}{2}.$$

So $y_0 = 1$, and $y_1 = 1 + \dfrac{t^2}{2}$.

The Proof of Picard-Lindelöf's Theorem.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = t\, y, \qquad y(0) = 1.$$

Solution: Integral equation: $y(t) = 1 + \int_0^t s\, y(s)\, ds$.

And $y_0 = 1$, and $y_1 = 1 + \frac{t^2}{2}$. Let's compute y_2,

$$y_2 = 1 + \int_0^t s\, y_1(s)\, ds = 1 + \int_0^t \Big(s + \frac{s^3}{2}\Big)\, ds = 1 + \frac{t^2}{2} + \frac{t^4}{8}.$$

So we've got $y_2(t) = 1 + \Big(\frac{t^2}{2}\Big) + \frac{1}{2}\Big(\frac{t^2}{2}\Big)^2$. Show that:

$$y_3(t) = 1 + \Big(\frac{t^2}{2}\Big) + \frac{1}{2!}\Big(\frac{t^2}{2}\Big)^2 + \frac{1}{3!}\Big(\frac{t^2}{2}\Big)^3.$$

The Proof of Picard-Lindelöf's Theorem.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = t\, y, \qquad y(0) = 1.$$

Solution: $y_3(t) = 1 + \Big(\frac{t^2}{2}\Big) + \frac{1}{2!}\Big(\frac{t^2}{2}\Big)^2 + \frac{1}{3!}\Big(\frac{t^2}{2}\Big)^3$.

By computng few more terms one finds

$$y_n(t) = \sum_{k=0}^{n} \frac{1}{k!}\Big(\frac{t^2}{2}\Big)^k.$$

Hence the limit $n \to \infty$ is given by

$$y(t) = \lim_{n\to\infty} y_n(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\frac{t^2}{2}\Big)^k = e^{t^2/2},$$

since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. We conclude, $y(t) = e^{t^2/2}$. $\triangleleft$

The Proof of Picard-Lindelöf's Theorem.

Sketch of the proof: Integrate on both sides with respect to t,

$$\int_{t_0}^{t} y'(s)\,ds = \int_{t_0}^{t} f(s,y(s))\,ds \;\Rightarrow\; y(t) = y_0 + \int_{t_0}^{t} f(s,y(s))\,ds,$$

Construct a sequence of continuous functions, $\{y_n\}_{n=0}^{\infty}$,

$$y_0(t) = y_0, \quad y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s,y_n(s))\,ds.$$

This is a Cauchy sequence in a small enough domain $D_b = [t_0 - b, t_0 + b]$. Introduce the norm on the space of continuous functions

$$\|u\| = \max_{t\in D_b} |u(t)|.$$

The Proof of Picard-Lindelöf's Theorem.

Sketch of the proof: Two consecutive elements in the sequence satisfy

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \max_{t\in D_b}\Big|\int_{t_0}^{t} f(s,y_n(s))\,ds - \int_{t_0}^{t} f(s,y_{n-1}(s))\,ds\Big| \\
&\leqslant \max_{t\in D_b}\int_{t_0}^{t} \big|f(s,y_n(s)) - f(s,y_{n-1}(s))\big|\,ds \\
&\leqslant k\,\max_{t\in D_b}\int_{t_0}^{t} |y_n(s) - y_{n-1}(s)|\,ds \\
&\leqslant kb\,\|y_n - y_{n-1}\|.
\end{aligned}$$

So we have,

$$\|y_{n+1} - y_n\| \leqslant r\,\|y_n - y_{n-1}\| \quad\Rightarrow\quad \|y_{n+1} - y_n\| \leqslant r^n\,\|y_1 - y_0\|.$$

The Proof of Picard-Lindelöf's Theorem.

Sketch of the proof: Recall: $\|y_{n+1} - y_n\| \leqslant r^n \|y_1 - y_0\|$.
Using the triangle inequality for norms and and the sum of a geometric series one compute the following,

$$\begin{aligned}
\|y_n - y_{n+m}\| &= \|y_n - y_{n+1} + y_{n+1} - y_{n+2} + \cdots + y_{n+(m-1)} - y_{n+m}\| \\
&\leqslant \|y_n - y_{n+1}\| + \|y_{n+1} - y_{n+2}\| + \cdots + \|y_{n+(m-1)} - y_{n+m}\| \\
&\leqslant (r^n + r^{n+1} + \cdots + r^{n+m}) \|y_1 - y_0\| \\
&\leqslant r^n(1 + r + r^2 + \cdots + r^m) \|y_1 - y_0\| \\
&\leqslant r^n \left(\frac{1 - r^m}{1 - r}\right) \|y_1 - y_0\|.
\end{aligned}$$

Choose b such that $b < \min\{a, 1/k\}$, hence $0 < r < 1$.
Then $\{y_n\}$ is a Cauchy sequence in the Banach space $C(D_b)$, with norm $\| \; \|$, hence converges.
Then $y = \lim_{n\to\infty} y_n$ exists and satisfy the differential eq. $\square$

On linear and non-linear equations. (Sect. 1.6).

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- **Direction Fields.**

Direction Fields.

Remarks:

- One does not need to solve a differential equation $y'(t) = f(t, y(t))$ to have a qualitative idea of the solution.
- Recall that $y'(t)$ represents the slope of the tangent line to the graph of function y at the point $(t, y(t))$.
- A differential equation provides these slopes, $f(t, y(t))$, for every point $(t, y(t))$.
- Key idea: Graph the function $f(t, y)$ on the yt-plane, not as points, but as slopes of small segments.

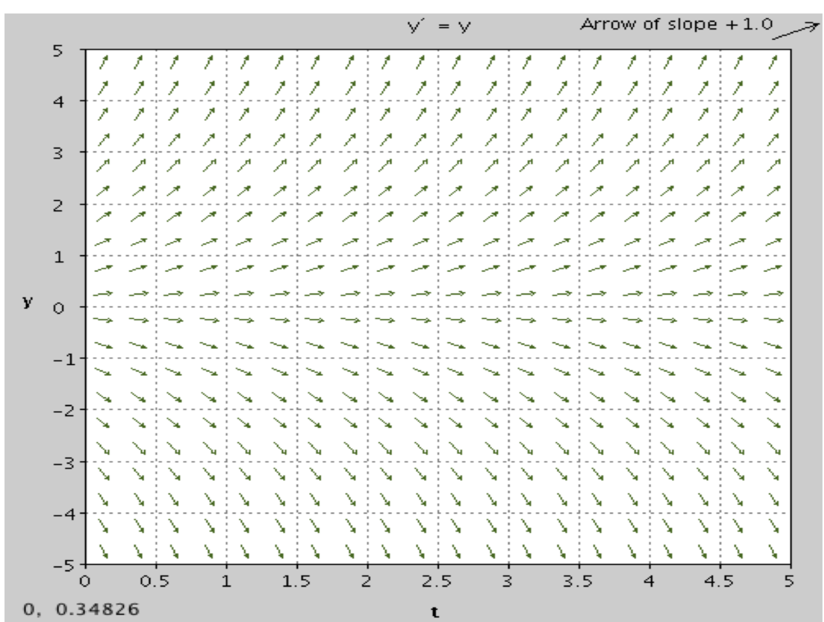
Definition

A *Direction Field* for the differential equation $y'(t) = f(t, y(t))$ is the graph on the yt-pane of the values $f(t, y)$ as slopes of a small segments.

Direction Fields.

Example

We know that the solution of $y' = y$ are the exponentials $y(t) = y_0\, e^t$. The graph of these solution is simple.
So is the direction field:



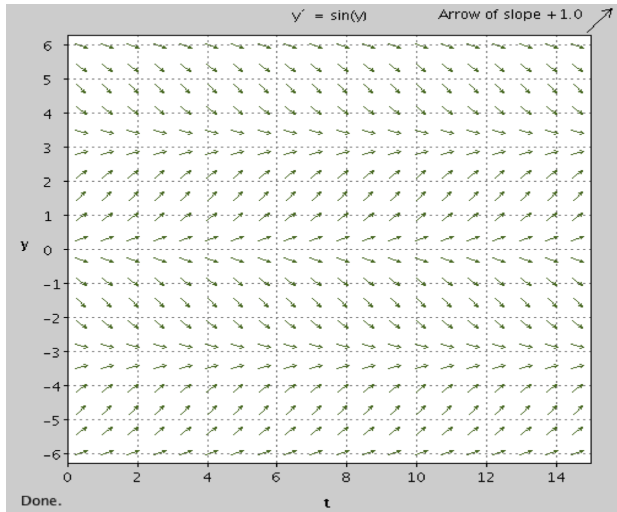
Direction Fields.

Example

The solution of $y' = \sin(y)$ is simple to compute. The equation is separable. After some calculations the implicit solution are

$$\ln\left|\frac{\csc(y_0)+\cot(y)}{\csc(y)+\cot(y)}\right| = t.$$

for $y_0 \in \mathbb{R}$. The graph of these solution is not simple to do.
But the direction field is simple to plot:



Direction Fields.

Example

The solution of $y' = \dfrac{(1+y^3)}{(1+t^2)}$ could be hard to compute. But the direction field is simple to plot: