Modeling with first order equations (Sect. 1.5).

- Radioactive decay.
  - Carbon-14 dating.
- Salt in a water tank.
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.

Radioactive decay

Remarks:

(a) Radioactive substances randomly emit protons, electors, radiation, and they are transformed in another substance.

(b) It can be seen that the time rate of change of the amount N of a radioactive substances is proportional to the negative amount of radioactive substance.

$$N'(t) = -a\,N(t), \qquad N(0) = N_0, \qquad a > 0.$$

(c) The integrating factor method implies $N(t) = N_0\, e^{-at}$.

(d) The *half-life* is the time τ needed to get $N(\tau) = N_0/2$.

$$N_0\, e^{-a\tau} = \frac{N_0}{2} \quad \Rightarrow \quad -a\tau = \ln\Big(\frac{1}{2}\Big) \quad \Rightarrow \quad \tau = \frac{\ln(2)}{a}.$$

(e) Using the half-life, we get $N(t) = N_0\, 2^{-t/\tau}$.

Radioactive decay

Example

Remains containing 14% of the original amount of Carbon-14 are found. Knowing that Carbon-14 half-live is $\tau = 5730$ years, date the remains.

Solution: Set $t = 0$ when the organism dies. Since the amount N of Carbon-14 only decays after the organism dies,

$$N(t) = N_0 \, 2^{-t/\tau}, \qquad \tau = 5730 \text{ years}.$$

The remains contain 14% of the original amount at the time t,

$$\frac{N(t)}{N_0} = \frac{14}{100} \quad \Rightarrow \quad 2^{-t/\tau} = \frac{14}{100}$$

$$-\frac{t}{\tau} = \log_2(14/100) \quad \Rightarrow \quad t = \tau \log_2(100/14).$$

The organism died 16,253 years ago. ◁

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Salt in a water tank.

Problem: Describe the salt concentration in a tank with water if salty water comes in and goes out of the tank.

Main ideas of the test:

- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- The amount of salt in the tank depends on the salt concentration coming in and going out of the tank.
- The salt in the tank also depends on the water rates coming in and going out of the tank.
- To construct a model means to find the differential equation that takes into account the above properties of the system.
- Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.

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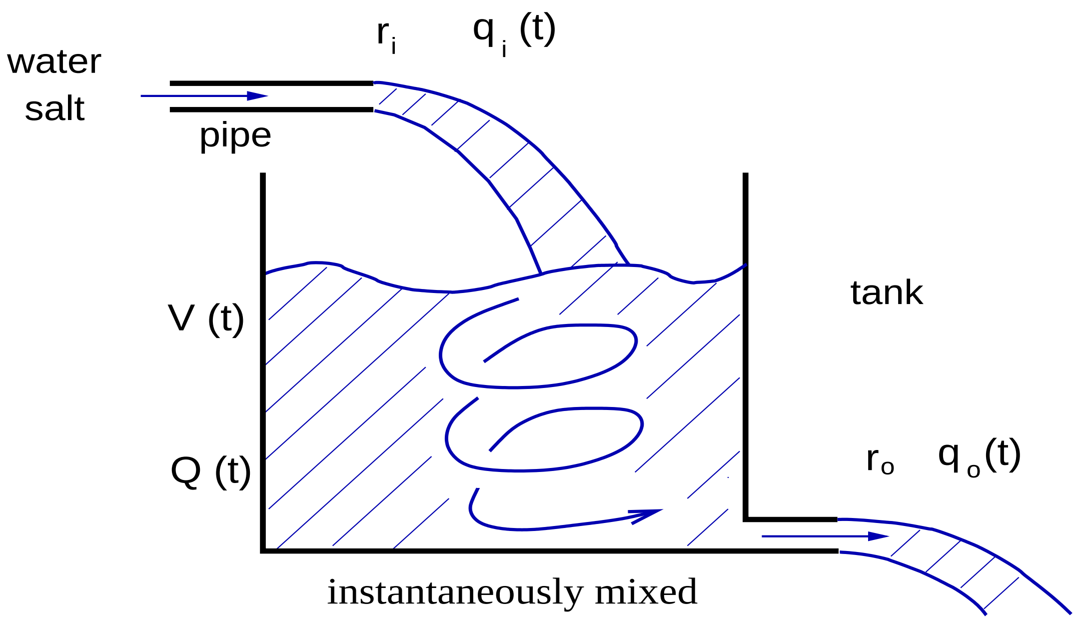
The experimental device.



The experimental device.

Definitions:

- $r_i(t)$, $r_o(t)$: Rates in and out of water entering and leaving the tank at the time t.
- $q_i(t)$, $q_o(t)$: Salt concentration of the water entering and leaving the tank at the time t.
- $V(t)$: Water volume in the tank at the time t.
- $Q(t)$: Salt mass in the tank at the time t.

Units:

$$\big[r_i(t)\big] = \big[r_o(t)\big] = \frac{\text{Volume}}{\text{Time}}, \qquad \big[q_i(t)\big] = \big[q_o(t)\big] = \frac{\text{Mass}}{\text{Volume}}.$$

$$\big[V(t)\big] = \text{Volume}, \qquad \big[Q(t)\big] = \text{Mass}.$$

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The main equations.

Remark: The mass conservation provides the main equations of the mathematical description for salt in water.

Main equations:

$$\frac{d}{dt}V(t) = r_i(t) - r_o(t), \qquad \text{Volume conservation,} \qquad (1)$$

$$\frac{d}{dt}Q(t) = r_i(t)\,q_i(t) - r_o(t)\,q_o(t), \qquad \text{Mass conservation,} \qquad (2)$$

$$q_o(t) = \frac{Q(t)}{V(t)}, \qquad \text{Instantaneously mixed,} \qquad (3)$$

$$r_i,\ r_o : \quad \text{Constants.} \qquad (4)$$

The main equations.

Remarks:

$$\left[\frac{dV}{dt}\right] = \frac{\text{Volume}}{\text{Time}} = \left[r_i - r_o\right],$$

$$\left[\frac{dQ}{dt}\right] = \frac{\text{Mass}}{\text{Time}} = \left[r_i q_i - r_o q_o\right],$$

$$\left[r_i q_i - r_o q_o\right] = \frac{\text{Volume}}{\text{Time}} \, \frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}.$$

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Analysis of the mathematical model.

Eqs. (**??**) and (**??**) imply

$$V(t) = (r_i - r_o)\,t + V_0, \tag{5}$$

where $V(0) = V_0$ is the initial volume of water in the tank.

Eqs. (**??**) and (**??**) imply

$$\frac{d}{dt}Q(t) = r_i\,q_i(t) - r_o\,\frac{Q(t)}{V(t)}. \tag{6}$$

Eqs. (**??**) and (**??**) imply

$$\frac{d}{dt}Q(t) = r_i\,q_i(t) - \frac{r_o}{(r_i - r_o)\,t + V_0}\,Q(t). \tag{7}$$

Analysis of the mathematical model.

Recall: $\displaystyle \frac{d}{dt}Q(t) = r_i\,q_i(t) - \frac{r_o}{(r_i - r_o)\,t + V_0}\,Q(t).$

Notation: $\displaystyle a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}$, and $b(t) = r_i\,q_i(t)$.

The main equation of the description is given by

$$Q'(t) = a(t)\,Q(t) + b(t).$$

Linear ODE for Q. Solution: Integrating factor method.

$$Q(t) = e^{A(t)}\left[Q_0 + \int_0^t e^{-A(s)}\,b(s)\,ds\right]$$

with $Q(0) = Q_0$, and $\displaystyle A(t) = \int_0^t a(s)\,ds.$

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Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.
If r, q_i, Q_0 and V_0 are given, find $Q(t)$.

Solution: Always holds $Q'(t) = a(t)\, Q(t) + b(t)$.
In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)\, t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i\, q_i(t) \quad \Rightarrow \quad b(t) = r q_i = b_0.$$

We need to solve the IVP:

$$Q'(t) = -a_0\, Q(t) + b_0, \quad Q(0) = Q_0.$$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.
If r, q_i, Q_0 and V_0 are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) + a_0\,Q(t) = b_0, \quad Q(0) = Q_0.$

Integrating factor method:

$$A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s}\, b_0\, ds.$$

$$Q(t) = e^{-a_0 t}\Big[Q_0 + \frac{b_0}{a_0}\big(e^{a_0 t} - 1\big)\Big]. = \Big(Q_0 - \frac{b_0}{a_0}\Big)\, e^{-a_0 t} + \frac{b_0}{a_0}.$$

But $\dfrac{b_0}{a_0} = r q_i\,\dfrac{V_0}{r} = q_i V_0$, and $a_0 = \dfrac{r}{V_0}$. We conclude:

$$Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0.$$

Predictions for particular situations.

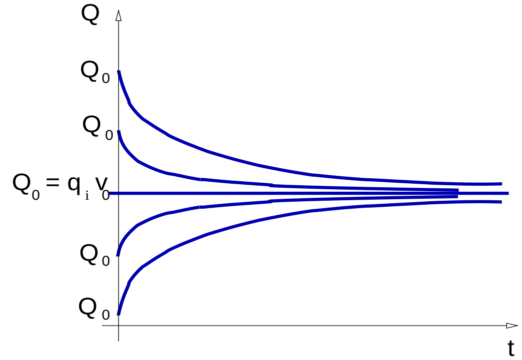
Example

Assume that $r_i = r_o = r$ and q_i are constants.
If r, q_i, Q_0 and V_0 are given, find $Q(t)$.

Solution: Recall: $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0.$

Particular cases:

- $\dfrac{Q_0}{V_0} > q_i$;
- $\dfrac{Q_0}{V_0} = q_i$, so $Q(t) = Q_0$;
- $\dfrac{Q_0}{V_0} < q_i$.



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Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o)\, t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.
So $q(t) = Q(t)/V(t)$ is given by $q(t) = \dfrac{Q_0}{V_0}\, e^{-rt/V_0}$. Therefore,

$$\frac{1}{100}\,\frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0}\, e^{-rt_1/V_0} \quad \Rightarrow \quad e^{-rt_1/V_0} = \frac{1}{100}.$$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\, t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0}\, t_1 = \ln(100).$$

We conclude that $t_1 = \dfrac{V_0}{r}\, \ln(100)$.

In this case: $t_1 = 100\, \ln(100)$. $\lhd$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ are constants. If $r = 5x10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

Solution: Recall: $Q'(t) = a(t)\,Q(t) + b(t)$. In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i\,q_i(t) \quad \Rightarrow \quad b(t) = r\big[2 + \sin(2t)\big].$$

We need to solve the IVP: $Q'(t) = -a_0\,Q(t) + b(t), \; Q(0) = 0.$

$$e^{a_0 t}Q(t) = \int_0^t e^{a_0 s}\,b(s)\,ds.$$

We conclude: $Q(t) = re^{-rt/V_0}\displaystyle\int_0^t e^{rs/V_0}\,\big[2 + \sin(2s)\big]\,ds.$