Review Exam 4, Chapters 7 and 10.

- Sections 7.1-7.6, 7.8, 10.1-10.5.
- 5 or 6 problems.
- 50 minutes.
  - Eigenvalue-Eigenfunction, boundary value probl. (10.1).
  - Fourier series expansions (10.2).
  - The Fourier Convergence Theorem (10.3).
  - Even and Odd functions and extension of functions (10.4).
  - The heat equation and on separation of variables (10.5).

Review Exam 4, Chapters 7 and 10.

- Sections 7.1-7.6, 7.8, 10.1-10.5.
- 5 or 6 problems.
- 50 minutes.
  - **Eigenvalue-Eigenfunction, boundary value probl. (10.1).**
  - Fourier series expansions (10.2).
  - The Fourier Convergence Theorem (10.3).
  - Even and Odd functions and extension of functions (10.4).
  - The heat equation and on separation of variables (10.5).

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$,

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$,

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0)$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8),$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8},$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8}, \quad \lambda = \left(\frac{n\pi}{8}\right)^2,$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8}, \quad \lambda = \left(\frac{n\pi}{8}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{8}\right), \quad n = 1, 2, \cdots \quad \triangleleft$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0)$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8),$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2},$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Then, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big[\frac{(2n+1)\pi}{16}\Big]^2,$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Then, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big[\frac{(2n+1)\pi}{16}\Big]^2, \quad y_n(x) = \sin\Big(\frac{(2n+1)\pi x}{16}\Big). \qquad \triangleleft$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$.

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$.

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$0 = y'(0)$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$0 = y'(0) = c_2$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1 \mu \sin(\mu x).$$

$$0 = y'(8) = c_1 \mu \sin(\mu 8),$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1 \mu \sin(\mu x).$$

$$0 = y'(8) = c_1 \mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi,$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi, \quad \Rightarrow \quad \mu = \frac{n\pi}{8}.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi, \quad \Rightarrow \quad \mu = \frac{n\pi}{8}.$$

Then, choosing $c_1 = 1$, for $n = 1, 2, \cdots$ holds

$$\lambda = \left(\frac{n\pi}{8}\right)^2,$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi, \quad \Rightarrow \quad \mu = \frac{n\pi}{8}.$$

Then, choosing $c_1 = 1$, for $n = 1, 2, \cdots$ holds

$$\lambda = \left(\frac{n\pi}{8}\right)^2, \quad y_n(x) = \cos\left(\frac{n\pi x}{8}\right).$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0)$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1,$$

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8)$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Then, choosing $c_1 = 1$, holds,

$$\lambda = 0,$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Then, choosing $c_1 = 1$, holds,

$$\lambda = 0, \qquad y_0(x) = 1.$$

$\triangleleft$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$.

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0)$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3)$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3)$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \;\Rightarrow\; c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \;\Rightarrow\; c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

$$c_1 = -\frac{\sqrt{3}/2}{1/2}$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \;\Rightarrow\; c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

$$c_1 = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}$$

Eigenvalue-Eigenfunction, boundary value problems

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \;\Rightarrow\; c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

$$c_1 = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3} \;\Rightarrow\; y(x) = -\sqrt{3}\cos(x) + \sin(x). \quad \triangleleft$$

Review Exam 4, Chapters 7 and 10.

- Sections 7.1-7.6, 7.8, 10.1-10.5.
- 5 or 6 problems.
- 50 minutes.
  - Eigenvalue-Eigenfunction, boundary value probl. (10.1).
  - **Fourier series expansions (10.2).**
  - **The Fourier Convergence Theorem (10.3).**
  - **Even, Odd functions and extensions (10.4).**
  - The heat equation and on separation of variables (10.5).

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1)\, \sin(n\pi x)\, dx$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1) \sin(n\pi x)\, dx = (-2)\, \frac{(-1)}{n\pi} \cos(n\pi x)\Big|_0^1,$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1)\, \sin(n\pi x)\, dx = (-2)\, \frac{(-1)}{n\pi}\, \cos(n\pi x)\Big|_0^1,$$

$$b_n = \frac{2}{n\pi} \big[\cos(n\pi) - 1 \big]$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1)\, \sin(n\pi x)\, dx = (-2)\, \frac{(-1)}{n\pi}\, \cos(n\pi x)\Big|_0^1,$$

$$b_n = \frac{2}{n\pi} \big[\cos(n\pi) - 1 \big] \quad \Rightarrow \quad b_n = \frac{2}{n\pi} \big[(-1)^n - 1 \big].$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \frac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[(-1)^{2k} - 1\big]$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \frac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\left[(-1)^n - 1\right]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\left[(-1)^{2k} - 1\right] = 0$.

If $n = 2k - 1$,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

If $n = 2k - 1$,

$$b_{(2k-1)} = \frac{2}{(2k-1)\pi}\big[(-1)^{2k-1} - 1\big]$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \frac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

If $n = 2k - 1$,

$$b_{(2k-1)} = \frac{2}{(2k-1)\pi}\big[(-1)^{2k-1} - 1\big] = -\frac{4}{(2k-1)\pi}.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\,\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\,\big[(-1)^{2k} - 1\big] = 0$.

If $n = 2k-1$,

$$b_{(2k-1)} = \frac{2}{(2k-1)\pi}\,\big[(-1)^{2k-1} - 1\big] = -\frac{4}{(2k-1)\pi}.$$

We conclude: $f(x) = -\dfrac{4}{\pi}\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin[(2k-1)\pi x]$. ◁

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx,$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \;\; L = 2,$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \;\; L = 2,$$

$$b_n = \int_0^2 (2 - x)\, \sin\Big(\frac{n\pi x}{2}\Big)\, dx.a$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx.$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx.$

$$\int \sin\left(\frac{n\pi x}{2}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right),$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2\int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx.$

$$\int \sin\left(\frac{n\pi x}{2}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$.

$$\int \sin\left(\frac{n\pi x}{2}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

$$I = \int x \sin\left(\frac{n\pi x}{2}\right) dx,$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2\int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$.

$$\int \sin\left(\frac{n\pi x}{2}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

$$I = \int x \sin\left(\frac{n\pi x}{2}\right) dx, \qquad \left\{ \begin{array}{l} u = x, \quad v' = \sin\left(\frac{n\pi x}{2}\right) \\ \\ \end{array} \right.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2\int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx.$

$$\int \sin\left(\frac{n\pi x}{2}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

$$I = \int x \sin\left(\frac{n\pi x}{2}\right) dx, \qquad \begin{cases} u = x, & v' = \sin\left(\frac{n\pi x}{2}\right) \\ u' = 1, & v = -\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \end{cases}$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2\int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\,dx - \int_0^2 x\,\sin\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$\int \sin\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{-2}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x\sin\Big(\frac{n\pi x}{2}\Big)\,dx, \qquad \begin{cases} u = x, & v' = \sin\Big(\dfrac{n\pi x}{2}\Big) \\ u' = 1, & v = -\dfrac{2}{n\pi}\,\cos\Big(\dfrac{n\pi x}{2}\Big) \end{cases}$$

$$I = \frac{-2x}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big) - \int\Big(\frac{-2}{n\pi}\Big)\,\cos\Big(\frac{n\pi x}{2}\Big)\,dx.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) dx.$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$$I = -\frac{2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big) + \Big(\frac{2}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{2}\Big).$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) dx.$

$I = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right).$ So, we get

$$b_n = 2\left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 + \left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$I = -\dfrac{2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) + \Big(\dfrac{2}{n\pi}\Big)^2 \sin\Big(\dfrac{n\pi x}{2}\Big).$ So, we get

$$b_n = 2\Big[\frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 + \Big[\frac{2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \Big[\frac{4}{n\pi}\cos(n\pi) - 0\Big]$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) dx.$

$I = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right).$ So, we get

$$b_n = 2\left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\bigg|_0^2 + \left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\bigg|_0^2 - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\bigg|_0^2$$

$$b_n = \frac{-4}{n\pi}\left[\cos(n\pi) - 1\right] + \left[\frac{4}{n\pi}\cos(n\pi) - 0\right] \quad \Rightarrow \quad b_n = \frac{4}{n\pi}.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$I = -\dfrac{2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) + \Big(\dfrac{2}{n\pi}\Big)^2 \sin\Big(\dfrac{n\pi x}{2}\Big).$ So, we get

$$b_n = 2\Big[\frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 + \Big[\frac{2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \Big[\frac{4}{n\pi}\cos(n\pi) - 0\Big] \quad \Rightarrow \quad b_n = \frac{4}{n\pi}.$$

We conclude: $f(x) = \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n} \sin\Big(\dfrac{n\pi x}{2}\Big).$ ◁

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic,

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series,

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\,dx = \int_0^2 (2 - x)\,dx = \frac{\text{base x height}}{2}$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \;\Rightarrow\; a_0 = 2.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \quad \Rightarrow \quad a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \;\Rightarrow\; a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx,$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\,dx = \int_0^2 (2 - x)\,dx = \frac{\text{base x height}}{2} \;\Rightarrow\; a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L} \int_0^L f(x) \cos\Big(\frac{n\pi x}{L}\Big)\,dx, \;\; L = 2,$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \quad \Rightarrow \quad a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \;\; L = 2,$$

$$a_n = \int_0^2 (2 - x)\, \cos\left(\frac{n\pi x}{2}\right) dx.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$

$$\int \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right),$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2\int_0^2 \cos\Big(\frac{n\pi x}{2}\Big)\,dx - \int_0^2 x\,\cos\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$\int \cos\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{2}{n\pi}\,\sin\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2\int_0^2 \cos\Big(\frac{n\pi x}{2}\Big)\,dx - \int_0^2 x\,\cos\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$\int \cos\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{2}{n\pi}\,\sin\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x\cos\Big(\frac{n\pi x}{2}\Big)\,dx,$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\Big(\frac{n\pi x}{2}\Big)\, dx - \int_0^2 x \cos\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$\int \cos\Big(\frac{n\pi x}{2}\Big)\, dx = \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \cos\Big(\frac{n\pi x}{2}\Big)\, dx, \quad \left\{ \begin{array}{l} u = x, \quad v' = \cos\Big(\frac{n\pi x}{2}\Big) \\ \\ \end{array} \right.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$

$$\int \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

$$I = \int x \cos\left(\frac{n\pi x}{2}\right) dx, \quad \begin{cases} u = x, & v' = \cos\left(\frac{n\pi x}{2}\right) \\ u' = 1, & v = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \end{cases}$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$

$$\int \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

$$I = \int x \cos\left(\frac{n\pi x}{2}\right) dx, \quad \begin{cases} u = x, & v' = \cos\left(\frac{n\pi x}{2}\right) \\ u' = 1, & v = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \end{cases}$$

$$I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) dx.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) dx.$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) dx.$

$$I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right).$$

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) dx.$

$I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right).$ So, we get

$$a_n = 2\left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) dx.$

$$I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right).$$ So, we get

$$a_n = 2\left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$a_n = 0 - 0 - \frac{4}{n^2\pi^2}\left[\cos(n\pi) - 1\right]$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) - \int \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) + \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big).$$ So, we get

$$a_n = 2\Big[\frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big[\frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$a_n = 0 - 0 - \frac{4}{n^2\pi^2}\big[\cos(n\pi) - 1\big] \quad \Rightarrow \quad a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n].$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0, \;\; a_0 = 2, \;\; a_n = \dfrac{4}{n^2\pi^2}[1 - (-1)^n].$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0, \;\; a_0 = 2, \;\; a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \frac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right]$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0, \;\; a_0 = 2, \;\; a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n].$

If $n = 2k$, then $a_{2k} = \frac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \frac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

If $n = 2k - 1$,

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0, \;\; a_0 = 2, \;\; a_n = \dfrac{4}{n^2\pi^2}[1 - (-1)^n].$

If $n = 2k$, then $a_{2k} = \dfrac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

If $n = 2k - 1$, then we obtain

$$a_{(2k-1)} = \frac{4}{(2k-1)^2\pi^2}\left[1 - (-1)^{2k-1}\right]$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0, \;\; a_0 = 2, \;\; a_n = \dfrac{4}{n^2\pi^2}[1 - (-1)^n].$

If $n = 2k$, then $a_{2k} = \dfrac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

If $n = 2k - 1$, then we obtain

$$a_{(2k-1)} = \frac{4}{(2k-1)^2\pi^2}\left[1 - (-1)^{2k-1}\right] = \frac{8}{(2k-1)^2\pi^2}.$$

Even-periodic, odd-periodic extension of functions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \dfrac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \dfrac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

If $n = 2k - 1$, then we obtain

$$a_{(2k-1)} = \frac{4}{(2k-1)^2\pi^2}\left[1 - (-1)^{2k-1}\right] = \frac{8}{(2k-1)^2\pi^2}.$$

We conclude: $f(x) = 1 + \dfrac{8}{\pi^2}\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\left(\frac{(2k-1)\pi x}{2}\right)$. $\triangleleft$

Review Exam 4, Chapters 7 and 10.

- Sections 7.1-7.6, 7.8, 10.1-10.5.
- 5 or 6 problems.
- 50 minutes.
  - Eigenvalue-Eigenfunction, boundary value probl. (10.1).
  - Fourier series expansions (10.2).
  - The Fourier Convergence Theorem (10.3).
  - Even and Odd functions and extension of functions (10.4).
  - **The heat equation** and on separation of variables (10.5).

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$.

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4 w_n(x)\, \frac{dv_n}{dt}(t) = v_n(t)\, \frac{d^2 w_n}{dx^2}(x)$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv_n}{dt}(t) = v_n(t)\,\frac{d^2 w_n}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)}$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv_n}{dt}(t) = v_n(t)\,\frac{d^2 w_n}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv_n}{dt}(t) = v_n(t)\,\frac{d^2 w_n}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0,$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv_n}{dt}(t) = v_n(t)\,\frac{d^2 w_n}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv_n}{dt}(t) = v_n(t)\,\frac{d^2 w_n}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

We solve for v_n with the initial condition $v_n(0) = 1$.

$$e^{\frac{\lambda_n}{4} t}\, v_n'(t) + \frac{\lambda_n}{4}\, e^{\frac{\lambda_n}{4} t}\, v_n(t) = 0$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv_n}{dt}(t) = v_n(t)\,\frac{d^2 w_n}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

We solve for v_n with the initial condition $v_n(0) = 1$.

$$e^{\frac{\lambda_n}{4}t}\, v_n'(t) + \frac{\lambda_n}{4}\, e^{\frac{\lambda_n}{4}t}\, v_n(t) = 0 \quad \Rightarrow \quad \Big[e^{\frac{\lambda_n}{4}t}\, v_n(t)\Big]' = 0.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0.$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t},$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0)$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(2) = 0$.

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(2) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$.

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(2) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(2) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(2) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(2) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(2) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1,$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(2) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2)$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2),$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0,$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4} t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$,

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$.

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2,$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \quad w_n(x) = \sin\left(\frac{n\pi x}{2}\right),$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \quad w_n(x) = \sin\left(\frac{n\pi x}{2}\right), \quad v_n(t) = e^{-(\frac{n\pi}{4})^2 t}.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \quad w_n(x) = \sin\left(\frac{n\pi x}{2}\right), \quad v_n(t) = e^{-(\frac{n\pi}{4})^2 t}.$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\Big(\frac{n\pi x}{2}\Big).$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2]$,

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\left(\frac{n\pi x}{2}\right).$

The initial condition is $3 \, \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \, \sin\left(\frac{n\pi x}{2}\right).$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\Big(\frac{n\pi x}{2}\Big).$

The initial condition is $3 \, \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} c_n \, \sin\Big(\frac{n\pi x}{2}\Big).$

The orthogonality of the sine functions implies

$$\int_0^2 3\sin\Big(\frac{\pi x}{2}\Big) \, \sin\Big(\frac{m\pi x}{2}\Big) \, dx = \sum_{n=1}^{\infty} \int_0^2 c_n \, \sin\Big(\frac{n\pi x}{2}\Big) \, \sin\Big(\frac{m\pi x}{2}\Big) \, dx.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\Big(\frac{n\pi x}{2}\Big).$

The initial condition is $3 \, \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} c_n \, \sin\Big(\frac{n\pi x}{2}\Big).$

The orthogonality of the sine functions implies

$$\int_0^2 3\sin\Big(\frac{\pi x}{2}\Big) \, \sin\Big(\frac{m\pi x}{2}\Big) \, dx = \sum_{n=1}^{\infty} \int_0^2 c_n \, \sin\Big(\frac{n\pi x}{2}\Big) \, \sin\Big(\frac{m\pi x}{2}\Big) \, dx.$$

If $m \neq 1$, then $0 = c_m \, \frac{2}{2}$,

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\Big(\frac{n\pi x}{2}\Big).$

The initial condition is $3 \, \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} c_n \, \sin\Big(\frac{n\pi x}{2}\Big).$

The orthogonality of the sine functions implies

$$\int_0^2 3\sin\Big(\frac{\pi x}{2}\Big) \, \sin\Big(\frac{m\pi x}{2}\Big) \, dx = \sum_{n=1}^{\infty} \int_0^2 c_n \, \sin\Big(\frac{n\pi x}{2}\Big) \, \sin\Big(\frac{m\pi x}{2}\Big) \, dx.$$

If $m \neq 1$, then $0 = c_m \, \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$.

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n\, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$

The initial condition is $3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right).$

The orthogonality of the sine functions implies

$$\int_0^2 3\sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_0^2 c_n \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx.$$

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$. Therefore,

$$3\sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right)$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$

The initial condition is $3\,\sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right).$

The orthogonality of the sine functions implies

$$\int_0^2 3\sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{m\pi x}{2}\right)dx = \sum_{n=1}^{\infty}\int_0^2 c_n \sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi x}{2}\right)dx.$$

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$. Therefore,

$$3\sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad c_1 = 3.$$

Example: Solving a Heat Equation.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: We conclude that

$$u(t, x) = 3\, e^{-(\frac{\pi}{4})^2 t} \sin\Big(\frac{\pi x}{2}\Big).$$

Review Exam 4, Chapters 7 and 10.

- Sections 7.1-7.6, 7.8, 10.1-10.5.
- 5 or 6 problems.
- 50 minutes.
  - Eigenvalue-Eigenfunction, boundary value probl. (10.1).
  - Fourier series expansions (10.2).
  - The Fourier Convergence Theorem (10.3).
  - Even and Odd functions and extension of functions (10.4).
  - The heat equation and **on separation of variables (10.5).**

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2 \, \partial_x^2 u(t,x) = 0.$$

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2\, \partial_x^2 u(t,x) = 0.$$

Solution: If $u(t,x) = v(t)\, w(x)$,

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2\, \partial_x^2 u(t,x) = 0.$$

Solution: If $u(t,x) = v(t)\, w(x)$, then

$$\partial_t^2 u(t,x) = w(x)\, \partial_t^2 v(t),$$

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2\, \partial_x^2 u(t,x) = 0.$$

Solution: If $u(t,x) = v(t)\, w(x)$, then

$$\partial_t^2 u(t,x) = w(x)\, \partial_t^2 v(t), \quad \partial_x^2 u(t,x) = v(t)\, \partial_x^2 w(x).$$

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2 \, \partial_x^2 u(t,x) = 0.$$

Solution: If $u(t,x) = v(t)\, w(x)$, then

$$\partial_t^2 u(t,x) = w(x)\, \partial_t^2 v(t), \quad \partial_x^2 u(t,x) = v(t)\, \partial_x^2 w(x).$$

Therefore, $\left(t + \frac{x}{c}\right) w(x)\, \partial_t^2 v(t) = -k^2 \, v(t)\, \partial_x^2 w(x),$

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2\, \partial_x^2 u(t,x) = 0.$$

Solution: If $u(t,x) = v(t)\, w(x)$, then

$$\partial_t^2 u(t,x) = w(x)\, \partial_t^2 v(t), \quad \partial_x^2 u(t,x) = v(t)\, \partial_x^2 w(x).$$

Therefore, $\left(t + \frac{x}{c}\right) w(x)\, \partial_t^2 v(t) = -k^2\, v(t)\, \partial_x^2 w(x),$

$$\left(t + \frac{x}{c}\right) \frac{\partial_t^2 v(t)}{v(t)} = -k^2\, \frac{\partial_x^2 w(x)}{w(x)}.$$

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2 \, \partial_x^2 u(t,x) = 0.$$

Solution: If $u(t,x) = v(t)\, w(x)$, then

$$\partial_t^2 u(t,x) = w(x)\, \partial_t^2 v(t), \quad \partial_x^2 u(t,x) = v(t)\, \partial_x^2 w(x).$$

Therefore, $\left(t + \frac{x}{c}\right) w(x)\, \partial_t^2 v(t) = -k^2 \, v(t)\, \partial_x^2 w(x)$,

$$\left(t + \frac{x}{c}\right) \frac{\partial_t^2 v(t)}{v(t)} = -k^2 \, \frac{\partial_x^2 w(x)}{w(x)}.$$

Function of t and x

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2\, \partial_x^2 u(t,x) = 0.$$

Solution: If $u(t,x) = v(t)\, w(x)$, then

$$\partial_t^2 u(t,x) = w(x)\, \partial_t^2 v(t), \quad \partial_x^2 u(t,x) = v(t)\, \partial_x^2 w(x).$$

Therefore, $\left(t + \frac{x}{c}\right) w(x)\, \partial_t^2 v(t) = -k^2\, v(t)\, \partial_x^2 w(x),$

$$\left(t + \frac{x}{c}\right) \frac{\partial_t^2 v(t)}{v(t)} = -k^2\, \frac{\partial_x^2 w(x)}{w(x)}.$$

Function of t and x $=$ Function only of x

Not every PDE can be solved with SVM.

Example

Determine whether the Separation of Variables Method can be used to solve

$$\left(t + \frac{x}{c}\right) \partial_t^2 u(t,x) + k^2 \, \partial_x^2 u(t,x) = 0.$$

Solution: If $u(t,x) = v(t)\, w(x)$, then

$$\partial_t^2 u(t,x) = w(x)\, \partial_t^2 v(t), \quad \partial_x^2 u(t,x) = v(t)\, \partial_x^2 w(x).$$

Therefore, $\left(t + \frac{x}{c}\right) w(x)\, \partial_t^2 v(t) = -k^2 \, v(t)\, \partial_x^2 w(x)$,

$$\left(t + \frac{x}{c}\right) \frac{\partial_t^2 v(t)}{v(t)} = -k^2 \, \frac{\partial_x^2 w(x)}{w(x)}.$$

Function of t and x $\quad = \quad$ Function only of x

We conclude: The SVM can not be used in this equation. $\triangleleft$

Review Exam 4, Chapters 7 and 10.

- Sections 7.1-7.6, 7.8, 10.1-10.5.
- 5 or 6 problems.
- 50 minutes.
  - Eigenvalue-Eigenfunction, boundary value probl. (10.1).
  - Fourier series expansions (10.2).
  - The Fourier Convergence Theorem (10.3).
  - Even and Odd functions and extension of functions (10.4).
  - The heat equation and **on separation of variables (10.5).**

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0, x) = \sin(x), \quad \partial_t u(0, x) = 0, \quad u(t, 0) = 0, \quad u(t, 3\pi) = 0.$$

Remarks:

- The Wave Equation describes waves on a string,

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2 \, \partial_x^2 u, \; t > 0, \; x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Remarks:

- The Wave Equation describes waves on a string, waves in the ocean,

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Remarks:

- The Wave Equation describes waves on a string, waves in the ocean, sound in air, etc.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Remarks:

- The Wave Equation describes waves on a string, waves in the ocean, sound in air, etc.
- There are two initial conditions:

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Remarks:

- The Wave Equation describes waves on a string, waves in the ocean, sound in air, etc.
- There are two initial conditions:
  (1) Initial position of the string.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Remarks:

- The Wave Equation describes waves on a string, waves in the ocean, sound in air, etc.
- There are two initial conditions:
  (1) Initial position of the string.
  (2) Initial velocity of the string.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Remarks:

- The Wave Equation describes waves on a string, waves in the ocean, sound in air, etc.
- There are two initial conditions:
  (1) Initial position of the string.
  (2) Initial velocity of the string.
- There are two boundary conditions:

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Remarks:

- The Wave Equation describes waves on a string, waves in the ocean, sound in air, etc.
- There are two initial conditions:
  (1) Initial position of the string.
  (2) Initial velocity of the string.
- There are two boundary conditions:
  (1) The string is fixed at both the end points.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2 \, \partial_x^2 u, \; t > 0, \; x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Let $u_n(t,x) = v_n(t) \, w_n(x)$. Then

$$w_n(x) \, \frac{d^2 v_n}{dt^2}(t) = c^2 \, v_n(t) \, \frac{d^2 w_n}{dx^2}(x)$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$. Then

$$w_n(x)\, \frac{d^2 v_n}{dt^2}(t) = c^2\, v_n(t)\, \frac{d^2 w_n}{dx^2}(x) \;\Rightarrow\; \frac{v_n''(t)}{c^2\, v_n(t)} = \frac{w_n''(x)}{w_n(x)}$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\,w_n(x)$. Then

$$w_n(x)\,\frac{d^2 v_n}{dt^2}(t) = c^2\,v_n(t)\,\frac{d^2 w_n}{dx^2}(x) \ \Rightarrow\ \frac{v_n''(t)}{c^2\,v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\,w_n(x)$. Then

$$w_n(x)\,\frac{d^2 v_n}{dt^2}(t) = c^2\,v_n(t)\,\frac{d^2 w_n}{dx^2}(x) \;\Rightarrow\; \frac{v_n''(t)}{c^2\,v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n''(t) + \lambda_n c^2\,v_n(t) = 0,$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$. Then

$$w_n(x)\, \frac{d^2 v_n}{dt^2}(t) = c^2\, v_n(t)\, \frac{d^2 w_n}{dx^2}(x) \ \Rightarrow\ \frac{v_n''(t)}{c^2\, v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n''(t) + \lambda_n c^2\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$. Then

$$w_n(x)\, \frac{d^2 v_n}{dt^2}(t) = c^2\, v_n(t)\, \frac{d^2 w_n}{dx^2}(x) \;\Rightarrow\; \frac{v_n''(t)}{c^2\, v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n''(t) + \lambda_n c^2\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

We first find the solution w_n to the BVP:

$$w_n''(x) + \lambda_n\, w_n(x) = 0, \quad w_n(0) = 0, \quad w_n(3\pi) = 0.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0, \ \ w_n(0) = w_n(3\pi) = 0.$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2 \, \partial_x^2 u, \; t > 0, \; x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n \, w_n(x) = 0, \;\; w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0,\ \ w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0,\ \ w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0,\ \ w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2 \, \partial_x^2 u, \; t > 0, \; x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n \, w_n(x) = 0, \;\; w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0, \ \ w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1,$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0,\ \ w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1\, \cos(\mu_n x) + c_2\, \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2\, \sin(\mu_n x).$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0,\ \ w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1\,\cos(\mu_n x) + c_2\,\sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2\,\sin(\mu_n x).$$

$$0 = w_n(3\pi) = c_2\,\sin(3\pi\mu_n),$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0,\ \ w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(3\pi) = c_2 \sin(3\pi\mu_n), \quad c_2 \neq 0,$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n''(x) + \lambda_n\, w_n(x) = 0,\ \ w_n(0) = w_n(3\pi) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(3\pi) = c_2 \sin(3\pi\mu_n), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(3\pi\mu_n) = 0.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n(x) = c_2 \sin(\mu_n x)$ and $\sin(3\pi\mu_n) = 0$.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n(x) = c_2\,\sin(\mu_n x)$ and $\sin(3\pi\mu_n) = 0$.

Then, $3\pi\mu_n = n\pi$,

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n(x) = c_2 \sin(\mu_n x)$ and $\sin(3\pi\mu_n) = 0$.

Then, $3\pi\mu_n = n\pi$, that is, $\mu_n = \dfrac{n}{3}$.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n(x) = c_2\,\sin(\mu_n x)$ and $\sin(3\pi\mu_n) = 0$.

Then, $3\pi\mu_n = n\pi$, that is, $\mu_n = \dfrac{n}{3}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \Big(\frac{n}{3}\Big)^2,$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n(x) = c_2\, \sin(\mu_n x)$ and $\sin(3\pi\mu_n) = 0$.

Then, $3\pi\mu_n = n\pi$, that is, $\mu_n = \frac{n}{3}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \Big(\frac{n}{3}\Big)^2, \qquad w_n(x) = \sin\Big(\frac{nx}{3}\Big).$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2 \, \partial_x^2 u, \; t > 0, \; x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n(x) = c_2 \sin(\mu_n x)$ and $\sin(3\pi\mu_n) = 0$.

Then, $3\pi\mu_n = n\pi$, that is, $\mu_n = \dfrac{n}{3}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n}{3}\right)^2, \qquad w_n(x) = \sin\left(\frac{nx}{3}\right).$$

We now find the general solution of $v_n''(t) + c^2 \mu_n^2 \, v_n(t) = 0$.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: Recall: $w_n(x) = c_2 \sin(\mu_n x)$ and $\sin(3\pi\mu_n) = 0$.

Then, $3\pi\mu_n = n\pi$, that is, $\mu_n = \dfrac{n}{3}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n}{3}\right)^2, \qquad w_n(x) = \sin\left(\frac{nx}{3}\right).$$

We now find the general solution of $v_n''(t) + c^2\mu_n^2\, v_n(t) = 0$.

Then the solution to the Wave Equation will be

$$u(t,x) = \sum_{n=1}^{\infty} v_n(t)\, \sin\left(\frac{nx}{3}\right).$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2 \, \partial_x^2 u, \; t > 0, \; x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution:

Recall: $\lambda_m = \left(\frac{n}{3}\right)^2, \;\; w_n(x) = \sin\left(\frac{nx}{3}\right), \;\; v_n''(t) + c^2 \mu_n^2 \, v_n(t) = 0.$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution:

Recall: $\lambda_m = \left(\frac{n}{3}\right)^2,\ w_n(x) = \sin\left(\frac{nx}{3}\right),\ v_n''(t) + c^2\mu_n^2\,v_n(t) = 0.$

$$v_n(t) = e^{rt}$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution:

Recall: $\lambda_m = \left(\frac{n}{3}\right)^2,\ w_n(x) = \sin\left(\frac{nx}{3}\right),\ v_n''(t) + c^2\mu_n^2\, v_n(t) = 0.$

$$v_n(t) = e^{rt} \quad \Rightarrow \quad p(r) = r^2 + c^2\mu_n^2 = 0$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2 \, \partial_x^2 u, \; t > 0, \; x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution:

Recall: $\lambda_m = \left(\frac{n}{3}\right)^2, \; w_n(x) = \sin\left(\frac{nx}{3}\right), \; v_n''(t) + c^2 \mu_n^2 \, v_n(t) = 0.$

$$v_n(t) = e^{rt} \quad \Rightarrow \quad p(r) = r^2 + c^2 \mu_n^2 = 0 \quad \Rightarrow \quad r_\pm = \pm c\mu_n i.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution:

Recall: $\lambda_m = \left(\frac{n}{3}\right)^2, \ w_n(x) = \sin\left(\frac{nx}{3}\right), \ v_n''(t) + c^2 \mu_n^2\, v_n(t) = 0.$

$$v_n(t) = e^{rt} \quad \Rightarrow \quad p(r) = r^2 + c^2 \mu_n^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm c\mu_n i.$$

A real-valued general solution is

$$v_n(t) = c_n \cos(c\mu_n t) + d_n \sin(c\mu_n t).$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution:

Recall: $\lambda_m = \left(\frac{n}{3}\right)^2$, $w_n(x) = \sin\left(\frac{nx}{3}\right)$, $v_n''(t) + c^2 \mu_n^2\, v_n(t) = 0$.

$$v_n(t) = e^{rt} \quad \Rightarrow \quad p(r) = r^2 + c^2 \mu_n^2 = 0 \quad \Rightarrow \quad r_\pm = \pm c\mu_n i.$$

A real-valued general solution is

$$v_n(t) = c_n \cos(c\mu_n t) + d_n \sin(c\mu_n t).$$

$$u(t,x) = \sum_{n=1}^{\infty} \left[c_n \cos(c\mu_n t) + d_n \sin(c\mu_n t)\right] \sin\left(\frac{nx}{3}\right).$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\; t > 0,\; x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big)\Big] \sin\Big(\frac{nx}{3}\Big).$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \displaystyle\sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big)\Big] \sin\Big(\frac{nx}{3}\Big).$

First initial condition: $\sin(x) = u(0,x)$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \displaystyle\sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big)\Big] \sin\Big(\frac{nx}{3}\Big).$

First initial condition: $\sin(x) = u(0,x) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\Big(\frac{nx}{3}\Big).$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big)\Big] \sin\Big(\frac{nx}{3}\Big).$

First initial condition: $\sin(x) = u(0,x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin(mx/3)$ on both sides above and integrate

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \displaystyle\sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big)\Big] \sin\Big(\frac{nx}{3}\Big).$

First initial condition: $\sin(x) = u(0,x) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin(mx/3)$ on both sides above and integrate

$$\int_0^{3\pi} \sin(x)\, \sin\Big(\frac{mx}{3}\Big)\, dx = \sum_{n=1}^{\infty} c_n \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big)\, \sin\Big(\frac{mx}{3}\Big)\, dx.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \displaystyle\sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big)\Big] \sin\Big(\frac{nx}{3}\Big).$

First initial condition: $\sin(x) = u(0,x) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin(mx/3)$ on both sides above and integrate

$$\int_0^{3\pi} \sin(x)\, \sin\Big(\frac{mx}{3}\Big)\, dx = \sum_{n=1}^{\infty} c_n \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big)\, \sin\Big(\frac{mx}{3}\Big)\, dx.$$

The orthogonality of the sine functions implies:

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big) \Big] \sin\Big(\frac{nx}{3}\Big).$

First initial condition: $\sin(x) = u(0,x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin(mx/3)$ on both sides above and integrate

$$\int_0^{3\pi} \sin(x) \sin\Big(\frac{mx}{3}\Big)\,dx = \sum_{n=1}^{\infty} c_n \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big) \sin\Big(\frac{mx}{3}\Big)\,dx.$$

The orthogonality of the sine functions implies: If $m \neq 3$, then

$0 = c_m \frac{3\pi}{2}$,

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big)\Big] \sin\Big(\frac{nx}{3}\Big).$

First initial condition: $\sin(x) = u(0,x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin(mx/3)$ on both sides above and integrate

$$\int_0^{3\pi} \sin(x)\, \sin\Big(\frac{mx}{3}\Big)\, dx = \sum_{n=1}^{\infty} c_n \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big)\, \sin\Big(\frac{mx}{3}\Big)\, dx.$$

The orthogonality of the sine functions implies: If $m \neq 3$, then $0 = c_m\, \frac{3\pi}{2}$, that is, $c_m = 0$ for $m \neq 3$.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \displaystyle\sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big) \Big] \sin\Big(\frac{nx}{3}\Big).$

First initial condition: $\sin(x) = u(0,x) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin(mx/3)$ on both sides above and integrate

$$\int_0^{3\pi} \sin(x)\, \sin\Big(\frac{mx}{3}\Big)\, dx = \sum_{n=1}^{\infty} c_n \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big)\, \sin\Big(\frac{mx}{3}\Big)\, dx.$$

The orthogonality of the sine functions implies: If $m \neq 3$, then $0 = c_m \frac{3\pi}{2}$, that is, $c_m = 0$ for $m \neq 3$. Hence, $\sin(x) = c_3 \sin\Big(\frac{3x}{3}\Big)$,

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: $u(t,x) = \sum_{n=1}^{\infty} \Big[c_n \cos\Big(\frac{cnt}{3}\Big) + d_n \sin\Big(\frac{cnt}{3}\Big)\Big] \sin\Big(\frac{nx}{3}\Big)$.

First initial condition: $\sin(x) = u(0,x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{nx}{3}\Big)$.

Multiply by $\sin(mx/3)$ on both sides above and integrate

$$\int_0^{3\pi} \sin(x)\, \sin\Big(\frac{mx}{3}\Big)\, dx = \sum_{n=1}^{\infty} c_n \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big)\, \sin\Big(\frac{mx}{3}\Big)\, dx.$$

The orthogonality of the sine functions implies: If $m \neq 3$, then $0 = c_m\,\frac{3\pi}{2}$, that is, $c_m = 0$ for $m \neq 3$. Hence, $\sin(x) = c_3 \sin\Big(\frac{3x}{3}\Big)$, and then $c_3 = 1$.

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0, x) = \sin(x), \quad \partial_t u(0, x) = 0, \quad u(t, 0) = 0, \quad u(t, 3\pi) = 0.$$

Solution: After the first initial condition we get

$$u(t, x) = \cos(ct)\, \sin(x) + \sum_{n=1}^{\infty} d_n \sin\Big(\frac{cnt}{3}\Big) \sin\Big(\frac{nx}{3}\Big).$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: After the first initial condition we get

$$u(t,x) = \cos(ct)\,\sin(x) + \sum_{n=1}^{\infty} d_n \sin\Big(\frac{cnt}{3}\Big)\,\sin\Big(\frac{nx}{3}\Big).$$

Second initial condition: $0 = \partial_t u(0,x)$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: After the first initial condition we get

$$u(t,x) = \cos(ct)\,\sin(x) + \sum_{n=1}^{\infty} d_n \,\sin\Big(\frac{cnt}{3}\Big)\,\sin\Big(\frac{nx}{3}\Big).$$

Second initial condition: $0 = \partial_t u(0,x) = \sum_{n=1}^{\infty} \frac{cnd_n}{3}\,\sin\Big(\frac{nx}{3}\Big).$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: After the first initial condition we get

$$u(t,x) = \cos(ct)\,\sin(x) + \sum_{n=1}^{\infty} d_n \sin\Big(\frac{cnt}{3}\Big)\,\sin\Big(\frac{nx}{3}\Big).$$

Second initial condition: $0 = \partial_t u(0,x) = \sum_{n=1}^{\infty} \frac{cnd_n}{3} \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin\Big(\frac{mx}{3}\Big)$ on both sides above,

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\,\partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: After the first initial condition we get

$$u(t,x) = \cos(ct)\,\sin(x) + \sum_{n=1}^{\infty} d_n\,\sin\Big(\frac{cnt}{3}\Big)\,\sin\Big(\frac{nx}{3}\Big).$$

Second initial condition: $0 = \partial_t u(0,x) = \displaystyle\sum_{n=1}^{\infty} \frac{cnd_n}{3}\,\sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin\Big(\dfrac{mx}{3}\Big)$ on both sides above, then for $m = 1, 2, \cdots,$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: After the first initial condition we get

$$u(t,x) = \cos(ct)\, \sin(x) + \sum_{n=1}^{\infty} d_n\, \sin\Big(\frac{cnt}{3}\Big)\, \sin\Big(\frac{nx}{3}\Big).$$

Second initial condition: $0 = \partial_t u(0,x) = \sum_{n=1}^{\infty} \frac{cnd_n}{3}\, \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin\Big(\frac{mx}{3}\Big)$ on both sides above, then for $m = 1, 2, \cdots,$

$$0 = \sum_{n=1}^{\infty} \frac{cnd_n}{3} \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big)\, \sin\Big(\frac{mx}{3}\Big)\, dx$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi]$,

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: After the first initial condition we get

$$u(t,x) = \cos(ct)\, \sin(x) + \sum_{n=1}^{\infty} d_n \sin\Big(\frac{cnt}{3}\Big) \sin\Big(\frac{nx}{3}\Big).$$

Second initial condition: $0 = \partial_t u(0,x) = \displaystyle\sum_{n=1}^{\infty} \frac{cnd_n}{3} \sin\Big(\frac{nx}{3}\Big)$.

Multiply by $\sin\Big(\dfrac{mx}{3}\Big)$ on both sides above, then for $m = 1, 2, \cdots$,

$$0 = \sum_{n=1}^{\infty} \frac{cnd_n}{3} \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big) \sin\Big(\frac{mx}{3}\Big)\, dx \quad \Rightarrow \quad d_m = 0.$$

Example: Solving a Wave Equation.

Example

Find the solution to the IBVP $\partial_t^2 u = c^2\, \partial_x^2 u,\ t > 0,\ x \in [0, 3\pi],$

$$u(0,x) = \sin(x), \quad \partial_t u(0,x) = 0, \quad u(t,0) = 0, \quad u(t,3\pi) = 0.$$

Solution: After the first initial condition we get

$$u(t,x) = \cos(ct)\,\sin(x) + \sum_{n=1}^{\infty} d_n \sin\Big(\frac{cnt}{3}\Big)\,\sin\Big(\frac{nx}{3}\Big).$$

Second initial condition: $0 = \partial_t u(0,x) = \sum_{n=1}^{\infty} \frac{cnd_n}{3} \sin\Big(\frac{nx}{3}\Big).$

Multiply by $\sin\Big(\frac{mx}{3}\Big)$ on both sides above, then for $m = 1, 2, \cdots,$

$$0 = \sum_{n=1}^{\infty} \frac{cnd_n}{3} \int_0^{3\pi} \sin\Big(\frac{nx}{3}\Big)\,\sin\Big(\frac{mx}{3}\Big)\,dx \quad \Rightarrow \quad d_m = 0.$$

We conclude: $u(t,x) = \cos(ct)\,\sin(x).$ $\triangleleft$

Review for Final Exam. Chapters 7, 6, 5.

- Systems of linear Equations (Chptr. 7).
- Laplace transforms (Chptr. 6).
- Power series solutions (Chptr. 5).
- Second order linear equations (Chptr. 3).
- First order differential equations (Chptr. 2).

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real,

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent,

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex,

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$,

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm i e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm ie^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm ie^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

$$\mathbf{x}^{(1)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big],$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm i e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

$$\mathbf{x}^{(1)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big],$$

$$\mathbf{x}^{(2)} = e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real,

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent,

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real,

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$,

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$.

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t},$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t}, \quad \mathbf{x}^{(2)} = (\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t}, \quad \mathbf{x}^{(2)} = (\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Then, the general solution is

$$\mathbf{x} = c_1\,\mathbf{v}\,e^{\lambda t} + c_2\,(\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix}$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0$$

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

Solution: Recall: $\lambda_{\pm} = \pm 3$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_\pm = \pm 3, \ \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \ \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$
The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0)$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \;\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \;\; \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \;\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \;\; \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3,\ \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \ \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$
The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \;\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \;\; \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$
The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

We conclude: $\mathbf{x}(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$ $\triangleleft$

Review for Final Exam. Chapters 7, 6, 5.

- Systems of linear Equations (Chptr. 7).
- **Laplace transforms (Chptr. 6).**
- Power series solutions (Chptr. 5).
- Second order linear equations (Chptr. 3).
- First order differential equations (Chptr. 2).

Laplace transforms (Chptr. 6).

Summary:

- Main Properties:

Laplace transforms (Chptr. 6).

Summary:

- Main Properties:

$$\mathcal{L}\big[f^{(n)}(t)\big] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

Laplace transforms (Chptr. 6).

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \quad (13)$$

Laplace transforms (Chptr. 6).

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \quad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \quad (14)$$

Laplace transforms (Chptr. 6).

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \qquad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \qquad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \qquad (14)$$

- Convolutions:

Laplace transforms (Chptr. 6).

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \qquad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \qquad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \qquad (14)$$

- Convolutions:

$$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)].$$

Laplace transforms (Chptr. 6).

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \quad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \quad (14)$$

- Convolutions:

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)].$$

- Partial fraction decompositions, completing the squares.

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)]$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$,

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2 + 9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2 + 9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

$$\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2+9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

$$\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

$$\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)}$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)}$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9)+(bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\dfrac{s}{(s^2+9)} + \dfrac{2}{3}\,\dfrac{3}{(s^2+9)} + e^{-5s}\,\dfrac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9},$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\dfrac{s}{(s^2+9)} + \dfrac{2}{3}\,\dfrac{3}{(s^2+9)} + e^{-5s}\,\dfrac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9}, \quad c = 0,$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\dfrac{s}{(s^2+9)} + \dfrac{2}{3}\,\dfrac{3}{(s^2+9)} + e^{-5s}\,\dfrac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9}, \quad c = 0, \quad b = -a$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9}, \quad c = 0, \quad b = -a \quad \Rightarrow \quad b = -\frac{1}{9}.$$

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big]$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \dfrac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(e^{-5s}\,\mathcal{L}[u(t)] - e^{-5s}\,\mathcal{L}[\cos(3t)]\Big)$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(e^{-5s}\,\mathcal{L}[u(t)] - e^{-5s}\,\mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(e^{-5s}\,\mathcal{L}[u(t)] - e^{-5s}\,\mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution:

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution:

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Therefore, we conclude that,

$$y(t) = 3\cos(3t) + \frac{2}{3}\,\sin(3t) + \frac{u_5(t)}{9}\Big[1 - \cos(3(t-5))\Big].$$

◁

Review for Final Exam. Chapters 7, 6, 5.

- Systems of linear Equations (Chptr. 7).
- Laplace transforms (Chptr. 6).
- **Power series solutions (Chptr. 5).**
- Second order linear equations (Chptr. 3).
- First order differential equations (Chptr. 2).

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point,

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point,

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$.

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

(c) Euler equation: $(x - x_0)^2\,y'' + \alpha\,(x - x_0)\,y' + \beta\,y = 0$.

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

(c) Euler equation: $(x - x_0)^2\,y'' + \alpha\,(x - x_0)\,y' + \beta\,y = 0$.

Solutions: If $y(x) = |x - x_0|^r$,

Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

(c) Euler equation: $(x - x_0)^2\, y'' + \alpha\, (x - x_0)\, y' + \beta\, y = 0$.

Solutions: If $y(x) = |x - x_0|^r$, then r is solution of the indicial equation $p(r) = r(r-1) + \alpha r + \beta = 0$.

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2 \, y'' + \alpha \, (x - x_0) \, y' + \beta \, y = 0.$$

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\, (x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals,

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\, |x - x_0|^{r_1} + c_2\, |x - x_0|^{r_2}.$$

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\, (x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\, |x - x_0|^{r_1} + c_2\, |x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex,

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\,|x - x_0|^{r_1} + c_2\,|x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_\pm = \lambda \pm \mu i$.

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\, |x - x_0|^{r_1} + c_2\, |x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_\pm = \lambda \pm \mu i$. Then, the real-valued general solution is

$$y(x) = c_1\, |x - x_0|^{\lambda} \cos\bigl(\mu \ln |x - x_0|\bigr) + c_2\, |x - x_0|^{\lambda} \sin\bigl(\mu \ln |x - x_0|\bigr).$$

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\, |x - x_0|^{r_1} + c_2\, |x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_\pm = \lambda \pm \mu i$. Then, the real-valued general solution is

$$y(x) = c_1\, |x - x_0|^{\lambda} \cos\big(\mu \ln |x - x_0|\big) + c_2\, |x - x_0|^{\lambda} \sin\big(\mu \ln |x - x_0|\big).$$

(iii) If $r_1 = r_2 = r$, real,

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\, |x - x_0|^{r_1} + c_2\, |x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_\pm = \lambda \pm \mu i$. Then, the real-valued general solution is

$$y(x) = c_1\, |x - x_0|^{\lambda} \cos\big(\mu \ln |x - x_0|\big) + c_2\, |x - x_0|^{\lambda} \sin\big(\mu \ln |x - x_0|\big).$$

(iii) If $r_1 = r_2 = r$, real, then the general solution is

$$y(x) = \big(c_1 + c_2 \ln |x - x_0|\big)\, |x - x_0|^{r}.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)}$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)} \quad \Rightarrow \quad -3y = \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)}.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)} \quad \Rightarrow \quad -3y = \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)}.$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)}.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n\, x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n\, x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n\, x^{(n-1)} \quad \Rightarrow \quad -3y = \sum_{n=0}^{\infty} (-3n) a_n\, x^{(n-1)}.$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n\, x^{(n-2)}.$$

$$\sum_{n=0}^{\infty} n(n-1) a_n\, x^{(n-2)} + \sum_{n=0}^{\infty} (-3n) a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty} (-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty} (-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty} (-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0.$

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$m = n - 2$$

$$m \to n$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty} (-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty} (-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$m = n - 2$ $\qquad$ $m = n - 1$

$m \to n$ $\qquad$ $m \to n$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$m = n - 2$ $\qquad$ $m = n - 1$ $\qquad$ $m = n + 1$

$m \to n$ $\qquad$ $m \to n$ $\qquad$ $m \to n$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$m = n-2$	$m = n-1$	$m = n+1$
$m \to n$	$m \to n$	$m \to n$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^{n} + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\, x^{n} + \sum_{n=1}^{\infty} a_{n-1}\, x^{n} = 0.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty}a_{n-1}\,x^n = 0.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty} a_{n-1}\,x^n = 0.$$

$$(2)(1)a_2 + (-3)(1)a_1 +$$

$$\sum_{n=1}^{\infty}\big[(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1}\big]\,x^n = 0$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty} a_{n-1}\,x^n = 0.$$

$$(2)(1)a_2 + (-3)(1)a_1 +$$

$$\sum_{n=1}^{\infty}\big[(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1}\big]\,x^n = 0$$

We conclude: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1. \quad \triangleleft$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \dfrac{3}{2}\,a_1,$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \dfrac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\, a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\, a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\, a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain $a_3 = \frac{3}{2}\, a_1 - \frac{a_0}{6}$.

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain $a_3 = \frac{3}{2}\,a_1 - \frac{a_0}{6}$.

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain $a_3 = \frac{3}{2}\,a_1 - \frac{a_0}{6}$.

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y(x) = a_0 + a_1 x + \frac{3}{2}\,a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2}\, a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots$.

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2}\, a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots$.

$$y(x) = a_0\Big(1 - \frac{1}{6}\, x^3 + \cdots\Big) + a_1\Big(x + \frac{3}{2}\, x^2 + \frac{3}{2}\, x^3 + \cdots\Big),$$

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2}\, a_1 x^2 + \Big(\frac{3}{2} a_1 - \frac{a_0}{6}\Big)\, x^3 + \cdots$.

$$y(x) = a_0\Big(1 - \frac{1}{6}\, x^3 + \cdots\Big) + a_1\Big(x + \frac{3}{2}\, x^2 + \frac{3}{2}\, x^3 + \cdots\Big),$$

We conclude that:

$$y_1(x) = 1 - \frac{1}{6}\, x^3 + \cdots\ ,$$

$$y_2(x) = x + \frac{3}{2}\, x^2 + \frac{3}{2}\, x^3 + \cdots\ . \qquad \triangleleft$$

Review for Final Exam. Chapters 3, 2.

- Systems of linear Equations (Chptr. 7).
- Laplace transforms (Chptr. 6).
- Power series solutions (Chptr. 5).
- Second order linear equations (Chptr. 3).
- First order differential equations (Chptr. 2).

Review for Final Exam. Chapters 3, 2.

- Systems of linear Equations (Chptr. 7).
- Laplace transforms (Chptr. 6).
- Power series solutions (Chptr. 5).
- **Second order linear equations (Chptr. 3).**
- First order differential equations (Chptr. 2).

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$,

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\,y' + a_0\,y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real,

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is
$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex,

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_{\pm} = \alpha \pm \beta i$,

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t}$$

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t}\, \cos(\beta t),$$

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t}\, \cos(\beta t), \qquad y_2(t) = e^{\alpha t}\, \sin(\beta t).$$

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t}\cos(\beta t), \qquad y_2(t) = e^{\alpha t}\sin(\beta t).$$

If $r_1 = r_2 = r$, real,

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t}\cos(\beta t), \qquad y_2(t) = e^{\alpha t}\sin(\beta t).$$

If $r_1 = r_2 = r$, real, then the general solution is

$$y(t) = (c_1 + c_2\, t)\, e^{rt}.$$

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*.

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook.

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients:

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters:

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation,

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian,

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$,

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1' = -\frac{y_2 g}{W},$$

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1' = -\frac{y_2 g}{W}, \qquad u_2' = \frac{y_1 g}{W}.$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method.

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$,

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\,y'' - 4x\,y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\,y_1(x)$, and find an equation for v.

$$y_2 = x^2 v,$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv,$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\,y'' - 4x\,y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\,y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\,v' + (2x^2 - 8x^2 + 6x^2)\,v = 0.$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\,y'' - 4x\,y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\,y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\,v' + (2x^2 - 8x^2 + 6x^2)\,v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\, y_1.$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\, y_1.$$

Choose $c_1 = 0$, $c_2 = 1$.

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\, y_1.$$

Choose $c_1 = 0$, $c_2 = 1$. Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. $\lhd$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt},$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4 + 12}\big]$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_\pm = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big]$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t}$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_\pm = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

But this $y_p = k\,e^{-t}$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

But this $y_p = k\,e^{-t}$ is solution of the homogeneous equation.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

But this $y_p = k\,e^{-t}$ is solution of the homogeneous equation.

Then propose $y_p(t) = kt\,e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t},$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t}$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3 \;\Rightarrow\; k = -\frac{3}{4}.$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3 \;\Rightarrow\; k = -\frac{3}{4}.$$

We obtain: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

(4) Find the general solution:

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$.

(5) Impose the initial conditions.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$1 = y(0)$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2,$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0)$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

$$\left.\begin{aligned} c_1 + c_2 &= 1, \\ 3_1 - c_2 &= 1 \end{aligned}\right\}$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

$$\left.\begin{aligned} c_1 + c_2 &= 1, \\ 3_1 - c_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\, e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\, e^{3t} + c_2\, e^{-t} - \frac{3}{4} t\, e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix}1 & 1\\ 3 & -1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix} \;\Rightarrow\; \begin{bmatrix}c_1\\ c_2\end{bmatrix} = \frac{1}{-4}\begin{bmatrix}-1 & -1\\ -3 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2\\ 2\end{bmatrix}.$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$,

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, we obtain,

$$y(t) = \frac{1}{2}\left(e^{3t} + e^{-t}\right) - \frac{3}{4}t\,e^{-t}.$$

$\triangleleft$

Review for Final Exam. Chapters 3, 2.

- Systems of linear Equations (Chptr. 7).
- Laplace transforms (Chptr. 6).
- Power series solutions (Chptr. 5).
- Second order linear equations (Chptr. 3).
- **First order differential equations (Chptr. 2).**

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\, y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\, dt}$.

- Separable, non-linear equations: $h(y)\, y' = g(t)$.

First order differential equations (Chptr. 2).

Summary:

- **Linear**, first order equations: $y' + p(t)\, y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\, dt}$.

- **Separable**, non-linear equations: $h(y)\, y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\, dt$,

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\, y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\, dt}$.

- Separable, non-linear equations: $h(y)\, y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\, dt$, that is,

$$\int h(u)\, du = \int g(t)\, dt + c.$$

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\, y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\, dt}$.

- Separable, non-linear equations: $h(y)\, y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\, dt$, that is,

$$\int h(u)\, du = \int g(t)\, dt + c.$$

  The solution can be found in implicit of explicit form.

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\, y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\, dt}$.

- Separable, non-linear equations: $h(y)\, y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\, dt$, that is,

  $$\int h(u)\, du = \int g(t)\, dt + c.$$

  The solution can be found in implicit of explicit form.

- Homogeneous equations can be converted into separable equations.

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

- Separable, non-linear equations: $h(y)\,y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\,dt$, that is,

  $$\int h(u)\,du = \int g(t)\,dt + c.$$

  The solution can be found in implicit of explicit form.

- Homogeneous equations can be converted into separable equations.

  Read page 49 in the textbook.

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\, y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\, dt}$.

- Separable, non-linear equations: $h(y)\, y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\, dt$, that is,

  $$\int h(u)\, du = \int g(t)\, dt + c.$$

  The solution can be found in implicit of explicit form.

- Homogeneous equations can be converted into separable equations.

  Read page 49 in the textbook.

- No modeling problems from Sect. 2.3.

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook,

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x,y)\, y' + M(x,y) = 0.$$

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x,y)\,y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

  $$N(x,y)\,y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ,

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x,y)\, y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ, such that $N = \partial_y \psi$

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x,y)\, y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ, such that $N = \partial_y \psi$ and $M = \partial_x \psi$.

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x, y)\, y' + M(x, y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ, such that $N = \partial_y \psi$ and $M = \partial_x \psi$.

  The solution of the differential equation is

$$\psi\big(x, y(x)\big) = c.$$

First order differential equations (Chptr. 2).

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

First order differential equations (Chptr. 2).

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)

First order differential equations (Chptr. 2).

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)

First order differential equations (Chptr. 2).

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)\,y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\,y' = g(t)$.)

First order differential equations (Chptr. 2).

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)\,y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\,y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)

First order differential equations (Chptr. 2).

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)\,y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\,y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)
5. Exact equations.
   (Check one equation: $N\,y' + M = 0$, and $\partial_t N = \partial_y M$.)

First order differential equations (Chptr. 2).

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\, y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)
5. Exact equations.
   (Check one equation: $N\, y' + M = 0$, and $\partial_t N = \partial_y M$.)
6. Exact equation with integrating factor.
   (Very complicated to check.)

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number,

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example.

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)}$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.$$

$$v(x) = \frac{y}{x}$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$y = x\,v,$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

$$x\,v' = \frac{1 + v + v^2}{v} - v$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

$$x\,v' = \frac{1 + v + v^2}{v} - v = \frac{1 + v + v^2 - v^2}{v}$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

$$x\,v' = \frac{1 + v + v^2}{v} - v = \frac{1 + v + v^2 - v^2}{v} \quad \Rightarrow \quad x\,v' = \frac{1 + v}{v}.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$.

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1 + v}{v}$. This is a separable equation.

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1 + v(x)}\, v'(x) = \frac{1}{x}$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\,v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\,v'(x)\,dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$,

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c \quad \Rightarrow \quad 1 + v - \ln|1+v| = \ln|x| + c.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c \quad \Rightarrow \quad 1 + v - \ln|1+v| = \ln|x| + c.$$

$$v = \frac{y}{x}$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c \quad \Rightarrow \quad 1 + v - \ln|1+v| = \ln|x| + c.$$

$$v = \frac{y}{x} \quad \Rightarrow \quad 1 + \frac{y(x)}{x} - \ln\Big|1 + \frac{y(x)}{x}\Big| = \ln|x| + c. \qquad \triangleleft$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation,

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$,

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\,y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\,y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$,

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method. $\mu(x) = e^{-2x}$.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method. $\mu(x) = e^{-2x}$.

$$e^{-2x}\, v' - 2\, e^{-2x}\, v = 2$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\,y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\,y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\,v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method. $\mu(x) = e^{-2x}$.

$$e^{-2x}\,v' - 2\,e^{-2x}\,v = 2 \quad \Rightarrow \quad \big(e^{-2x}\,v\big)' = 2.$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$e^{-2x}\, v = 2x + c$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x}$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\,y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\big(e^{-2x}\,v\big)' = 2$.

$$e^{-2x}\,v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\,e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\,e^{2x}.$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2$.

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\, (2x + c)}$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2$.

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm\frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies:

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\, (2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x+c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x+c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2x\,(2x+c)} \quad\Rightarrow\quad y_{\pm}(x) = \pm\frac{e^{-x}}{\sqrt{2x+c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}}$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2$.

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \quad \Rightarrow \quad c = 9$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad\Rightarrow\quad y_{\pm}(x) = \pm\frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \quad\Rightarrow\quad c = 9 \quad\Rightarrow\quad y(x) = \frac{e^{-x}}{\sqrt{2x + 9}}. \quad \lhd$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$N = [2x^2y + 2x]$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$$

$$M = [2xy^2 + 2y]$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0$.

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$$

$$M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \ \partial_x N = \partial_y M.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0$.

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \;\Rightarrow\; \partial_x N = \partial_y M.$$

The equation is exact.

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2 y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2 y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2 y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2 y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N,$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \;\Rightarrow\; \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2 y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2 y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2 y + 2x$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \Rightarrow \ \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \;\Rightarrow\; \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x,y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2 y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2 y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2 y + 2x \quad \Rightarrow \quad \psi(x,y) = x^2 y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \;\Rightarrow\; \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \;\Rightarrow\; g'(x) = 0.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x,y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \;\Rightarrow\; g'(x) = 0.$$

$$\psi(x,y) = x^2y^2 + 2xy + c,$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N &= [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2. \\ M &= [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \Rightarrow \; \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x,y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \;\Rightarrow\; g'(x) = 0.$$

$$\psi(x,y) = x^2y^2 + 2xy + c, \quad x^2\, y^2(x) + 2x\, y(x) + c = 0. \quad \triangleleft$$