Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
- Main properties of even, odd functions.
- Sine and cosine series.
- Even-periodic, odd-periodic extensions of functions.

Even, odd functions.

Definition

A function $f : [-L, L] \to \mathbb{R}$ is *even* iff for all $x \in [-L, L]$ holds

$$f(-x) = f(x).$$

A function $f : [-L, L] \to \mathbb{R}$ is *odd* iff for all $x \in [-L, L]$ holds

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Even, odd functions.

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Remarks:

- The only function that is both odd and even is $f = 0$.

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Remarks:

- The only function that is both odd and even is $f = 0$.
- Most functions are neither odd nor even.

Even, odd functions.

Example

Show that the function $f(x) = x^2$ is even on $[-L, L]$.

Even, odd functions.

Example

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Solution: The function is even, since

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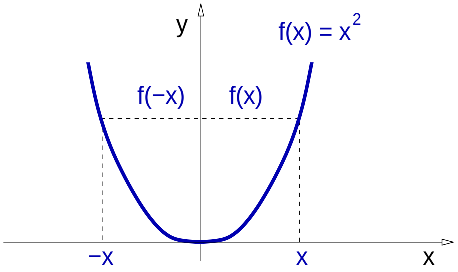

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Even, odd functions.

Example

Show that the function $f(x) = x^3$ is odd on $[-L, L]$.

Even, odd functions.

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Even, odd functions.

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Show that the function $f(x) = x^3$ is odd on $[-L, L]$.

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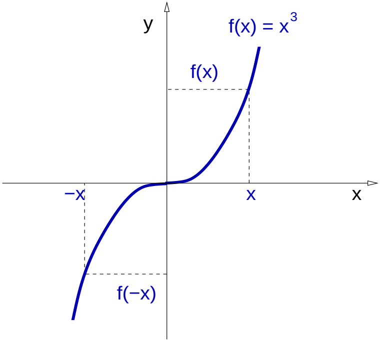

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Even, odd functions.

Example

(1) The function $f(x) = \cos(ax)$ is even on $[-L, L]$;

Even, odd functions.

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(1) The function $f(x) = \cos(ax)$ is even on $[-L, L]$;

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Even, odd functions.

Example

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(3) The functions $f(x) = e^x$ and $f(x) = (x-2)^2$ are neither even nor odd.

Even, odd functions.

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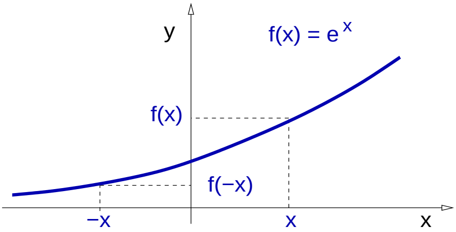

Even, odd functions.

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(1) The function $f(x) = \cos(ax)$ is even on $[-L, L]$;

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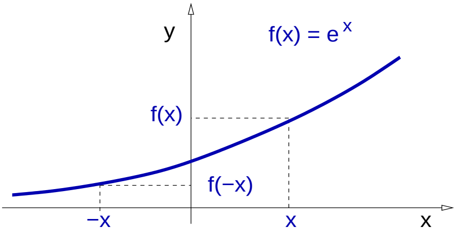

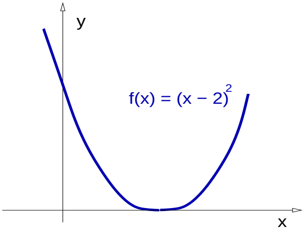

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Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
- **Main properties of even, odd functions.**
- Sine and cosine series.
- Even-periodic, odd-periodic extensions of functions.

Main properties of even, odd functions.

Theorem

(1) *A linear combination of even (odd) functions is even (odd).*

(2) *The product of two odd functions is even.*

(3) *The product of two even functions is even.*

(4) *The product of an even function by an odd function is odd.*

Main properties of even, odd functions.

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Proof:

(1) Let f and g be even,

Main properties of even, odd functions.

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Proof:

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Proof:

(1) Let f and g be even, that is, $f(-x) = f(x)$, $g(-x) = g(x)$. Then, for all $a, b \in \mathbb{R}$ holds,

$(af + bg)(-x)$

Main properties of even, odd functions.

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(1) Let f and g be even, that is, $f(-x) = f(x)$, $g(-x) = g(x)$.
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Case "odd" is similar.

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Cases (3), (4) are similar. $\square$

Main properties of even, odd functions.

Theorem

If $f : [-L, L] \to \mathbb{R}$ *is even, then* $\displaystyle\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$

If $f : [-L, L] \to \mathbb{R}$ *is odd, then* $\displaystyle\int_{-L}^{L} f(x)\,dx = 0.$

Main properties of even, odd functions.

Theorem

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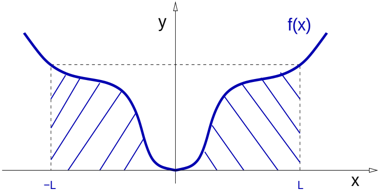
If $f : [-L, L] \to \mathbb{R}$ is odd, then $\displaystyle\int_{-L}^{L} f(x)\,dx = 0.$



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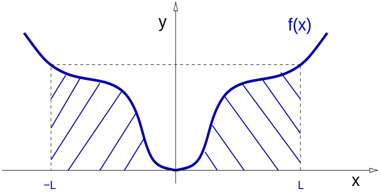

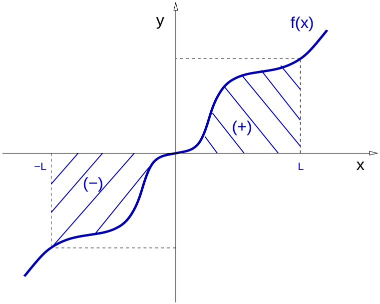

Main properties of even, odd functions.

Proof:

$$I = \int_{-L}^{L} f(x)\,dx$$

Main properties of even, odd functions.

Proof:

$$I = \int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx$$

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$$I = \int_{L}^{0} f(-y)\,(-dy) + \int_{0}^{L} f(x)\,dx$$

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Even case: $f(-y) = f(y)$,

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Even case: $f(-y) = f(y)$, therefore,

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Even case: $f(-y) = f(y)$, therefore,

$$I = \int_{0}^{L} f(y)\,dy + \int_{0}^{L} f(x)\,dx \;\Rightarrow\; \int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

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Odd case: $f(-y) = -f(y)$, therefore,

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Main properties of even, odd functions.

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$$I = \int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx \quad y = -x,\ dy = -dx.$$

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$$I = -\int_{0}^{L} f(y)\,dy + \int_{0}^{L} f(x)\,dx \quad\Rightarrow\quad \int_{-L}^{L} f(x)\,dx = 0. \qquad \square$$

Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
- Main properties of even, odd functions.
- **Sine and cosine series.**
- Even-periodic, odd-periodic extensions of functions.

Sine and cosine series.

Theorem (Cosine and Sine Series)

Consider the function $f : [-L, L] \to \mathbb{R}$ *with Fourier expansion*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

(1) *If* f *is even, then* $b_n = 0$ *for* $n = 1, 2, \cdots$, *and the Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big)$$

is called a Cosine Series.

(2) *If* f *is odd, then* $a_n = 0$ *for* $n = 0, 1, \cdots$, *and the Fourier series*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

is called a Sine Series.

Sine and cosine series.

Proof:

If f is even, and since the Sine function is odd,

Sine and cosine series.

Proof:

If f is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0,$$

Sine and cosine series.

Proof:

If f is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0,$$

since we are integrating an odd function on $[-L, L]$.

Sine and cosine series.

Proof:

If f is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0,$$

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If f is odd, and since the Cosine function is even,

Sine and cosine series.

Proof:
If f is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = 0,$$

since we are integrating an odd function on $[-L, L]$.

If f is odd, and since the Cosine function is even, then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx = 0,$$

Sine and cosine series.

Proof:

If f is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0,$$

since we are integrating an odd function on $[-L, L]$.

If f is odd, and since the Cosine function is even, then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 0,$$

since we are integrating an odd function on $[-L, L]$. $\square$

Sine and Cosine Series (Sect. 10.4).

- Even, odd functions.
- Main properties of even, odd functions.
- Sine and cosine series.
- **Even-periodic, odd-periodic extensions of functions.**

Even-periodic, odd-periodic extensions of functions.

(1) Even-periodic case:
A function $f : [0, L] \to \mathbb{R}$ can be extended as an even function $f : [-L, L] \to \mathbb{R}$ requiring for $x \in [0, L]$ that

$$f(-x) = f(x).$$

Even-periodic, odd-periodic extensions of functions.

(1) Even-periodic case:

A function $f : [0, L] \to \mathbb{R}$ can be extended as an even function $f : [-L, L] \to \mathbb{R}$ requiring for $x \in [0, L]$ that

$$f(-x) = f(x).$$

This function $f : [-L, L] \to \mathbb{R}$ can be further extended as a periodic function $f : \mathbb{R} \to \mathbb{R}$ requiring for $x \in [-L, L]$ that

$$f(x + 2nL) = f(x).$$

Even-periodic, odd-periodic extensions of functions.

Example

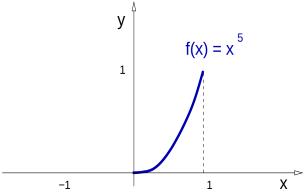
Sketch the graph of the even-periodic extension of $f(x) = x^5$, with $x \in [0, 1]$.

Even-periodic, odd-periodic extensions of functions.

Example

Sketch the graph of the even-periodic extension of $f(x) = x^5$, with $x \in [0, 1]$.

Solution:



Even-periodic, odd-periodic extensions of functions.

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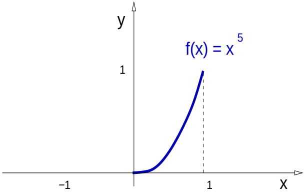

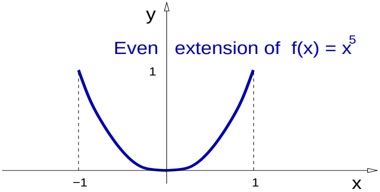

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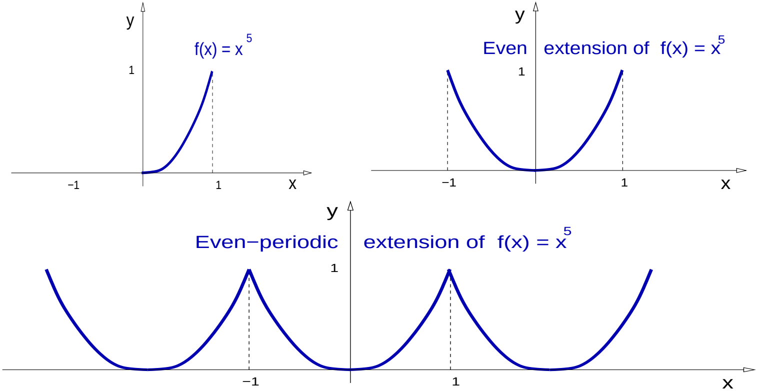

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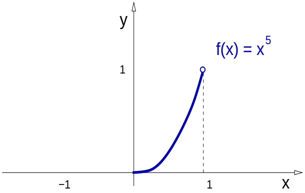

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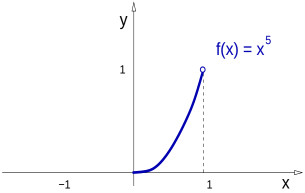

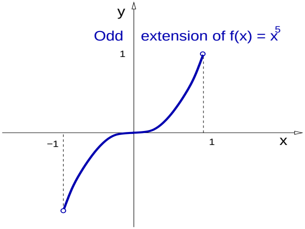

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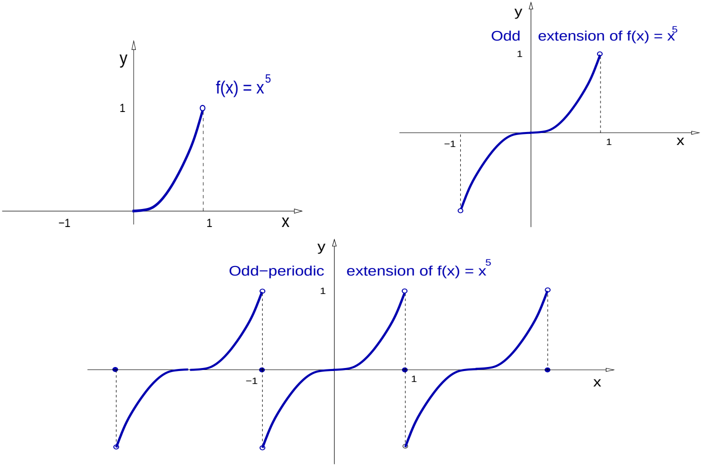

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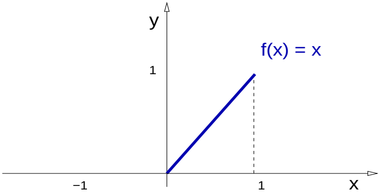

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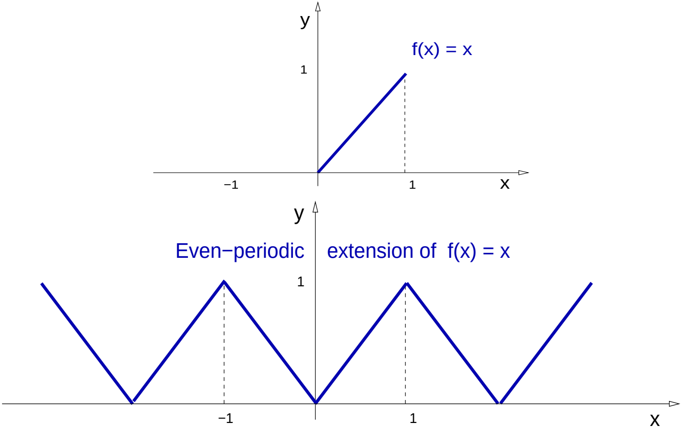

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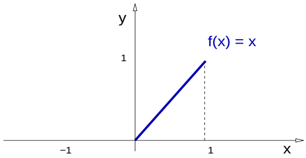

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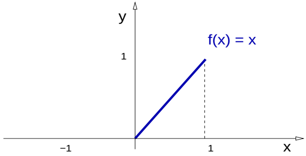

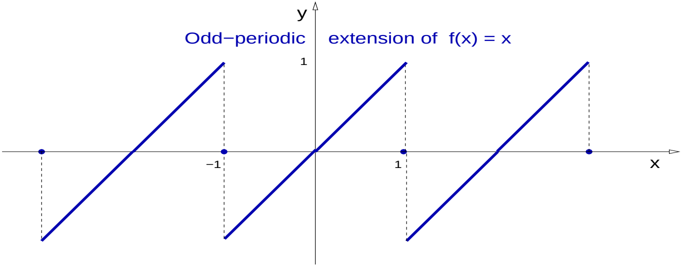

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Even-periodic, odd-periodic extensions of functions.

Example

Sketch the graph of the odd-periodic extension of $f(x) = x$, with $x \in (0, 1)$, and then find its Fourier Series.

Solution: Recall: $a_n = 0$, and $b_n = \dfrac{2(-1)^{n+1}}{n\pi}$.

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Sketch the graph of the odd-periodic extension of $f(x) = x$, with $x \in (0, 1)$, and then find its Fourier Series.

Solution: Recall: $a_n = 0$, and $b_n = \dfrac{2\,(-1)^{n+1}}{n\pi}$. Therefore,

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n} \sin(n\pi x). \qquad \triangleleft$$

Solving the Heat Equation (Sect. 10.5).

- Review: The Stationary Heat Equation.
- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
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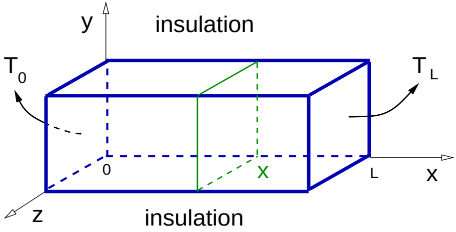

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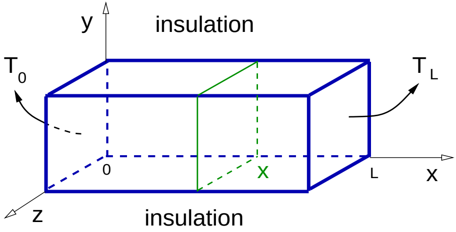

Remark: The heat transfer occurs only along the x-axis.

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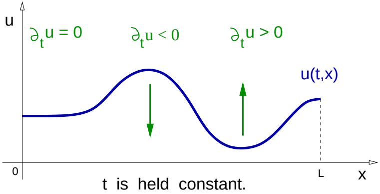

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Definition

The IBVP for the one-dimensional Heat Equation is the following: Given a constant $k > 0$ and a function $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \to \mathbb{R}$ solution of

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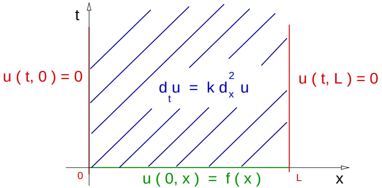

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The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

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Remark:
The separation of variables method does not work for every PDE.

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Summary:

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Here c_n are constants, $n = 1, 2, \cdots$.

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- We conclude that for appropriate constants λ_m holds

$$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = -\lambda_n,$$

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- We have transformed the original PDE into infinitely many ODEs parametrized by n, positive integer.

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Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

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The real-valued general solution is

$$w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x).$$

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We conclude that: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \cdots$.

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Summary: IBVP for the Heat Equation.

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$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.
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Remark:
The separation of variables method does not work for every PDE.

Solving the Heat Equation (Sect. 10.5).

- Review: The Stationary Heat Equation.
- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- **An example of separation of variables.**

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

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$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0,$$

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Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

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Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

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The orthogonality of the sine functions implies

$$3 \int_0^2 \sin\Big(\frac{\pi x}{2}\Big) \sin\Big(\frac{m\pi x}{2}\Big)\, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big) \sin\Big(\frac{m\pi x}{2}\Big)\, dx.$$

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$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\Big(\frac{n\pi x}{2}\Big).$

The initial condition is $3 \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{2}\Big).$

The orthogonality of the sine functions implies

$$3 \int_0^2 \sin\Big(\frac{\pi x}{2}\Big) \sin\Big(\frac{m\pi x}{2}\Big) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big) \sin\Big(\frac{m\pi x}{2}\Big) \, dx.$$

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$.

An example of separation of variables.

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$$3\sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad c_1 = 3.$$

An example of separation of variables.

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Solution: We conclude that

$$u(t, x) = 3\, e^{-(\frac{\pi}{4})^2 t} \sin\Big(\frac{\pi x}{2}\Big).$$

Review Exam 4.

- Sections 7.1-7.6, 7.8, 10.1-10.5.
- 5 or 6 problems.
- 50 minutes.
  - Overview of linear differential systems (7.1).
  - Review of Linear Algebra (7.2,7.3).
  - Basic Theorey of first order systems (7.4).
  - Homogeneous constant coefficients systems:
    - Real and different eigenvalues (7.5).
    - Complex eigenvalues (7.6).
    - Real and repeated eigenvalues (7.8).

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Example

Find the real-valued general solution of

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \qquad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

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Eigenvector for λ_+.

$$(A - \lambda_+ I) = \begin{bmatrix} 1 - (1+2i) & 2 \\ -2 & 1 - (1+2i) \end{bmatrix}$$

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$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \qquad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

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$$\mathbf{x}^{(1)} = e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(2t)\right)$$

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Hint to remember formulas for $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

Remark: The formulas for

$$\mathbf{x}^{(1)}(t) = e^{\alpha t}\big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big],$$

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are the real and imaginary part of $\tilde{\mathbf{x}}^{(+)} = (\mathbf{a} + \mathbf{b}i)\, e^{(\alpha+\beta i)t}$.

Hint to remember formulas for $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

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$$\tilde{\mathbf{x}}^{(+)} = (\mathbf{a} + \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big]\, e^{\alpha t}.$$

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Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

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Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

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Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

$\lambda^2 + 5\lambda + 4 = 0$

Exam: November 12, 2008. Problem 4.

Example

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Exam: November 12, 2008. Problem 4.

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Exam: November 12, 2008. Problem 4.

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Hence $\lambda_+ = -1$,

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Hence $\lambda_+ = -1$, $\lambda_- = -4$.

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Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for λ_+.

$$(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}$$

Exam: November 12, 2008. Problem 4.

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Exam: November 12, 2008. Problem 4.

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Exam: November 12, 2008. Problem 4.

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$2v_1 = \sqrt{2}\,v_2.$

Exam: November 12, 2008. Problem 4.

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Exam: November 12, 2008. Problem 4.

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Exam: November 12, 2008. Problem 4.

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$2v_1 = \sqrt{2}\,v_2$. Choosing $v_1 = \sqrt{2}$ and $v_2 = 2$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Exam: November 12, 2008. Problem 4.

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Exam: November 12, 2008. Problem 4.

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Eigenvector for λ_-.

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

Exam: November 12, 2008. Problem 4.

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$v_1 = -\sqrt{2}\, v_2$.

Exam: November 12, 2008. Problem 4.

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Exam: November 12, 2008. Problem 4.

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Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$,

Exam: November 12, 2008. Problem 4.

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Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$.

General solution: $\mathbf{x} = c_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$. ◁

Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

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Solution:

We start plotting the vectors

$$\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix},$$

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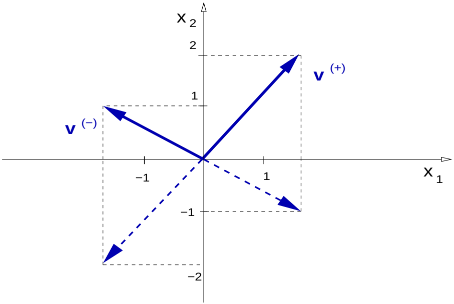

Exam: November 12, 2008. Problem 4.

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Solution:

We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

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Exam: November 12, 2008. Problem 4.

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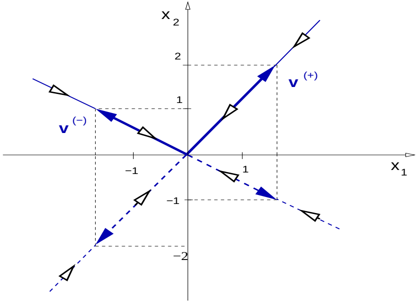

Exam: November 12, 2008. Problem 4.

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Solution:

Recall: $\lambda_- < \lambda_+ < 0$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)} e^{-t} + \mathbf{v}^{(-)} e^{-4t}.$$

Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

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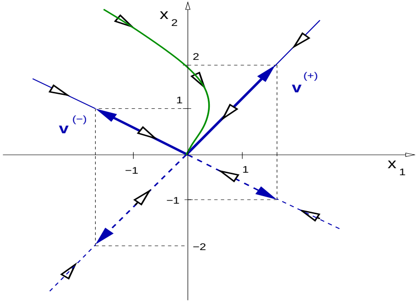

Exam: November 12, 2008. Problem 4.

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Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

Solution:
We plot the solutions

$$\mathbf{x} = c_1\, \mathbf{x}^{(+)} + c_2\, \mathbf{x}^{(-)},$$

for different values of c_1 and c_2.

Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

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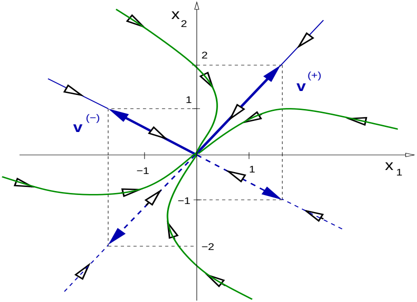

Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)} e^{\lambda_- t}$,

Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.
Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)} e^{\lambda_- t}$,

Solution:
Here $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

$$\mathbf{x}^{(-)}, \quad -\mathbf{x}^{(-)}.$$

Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

Solution:

Here $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

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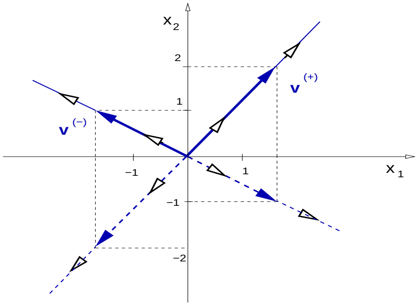

Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.
Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

Solution:
Recall: $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)}\, e^{4t} + \mathbf{v}^{(-)}\, e^{t}.$$

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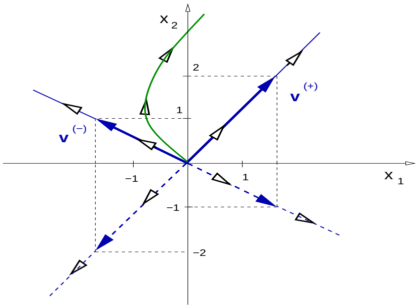

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$$\mathbf{x} = c_1 \mathbf{x}^{(+)} + c_2 \mathbf{x}^{(-)},$$

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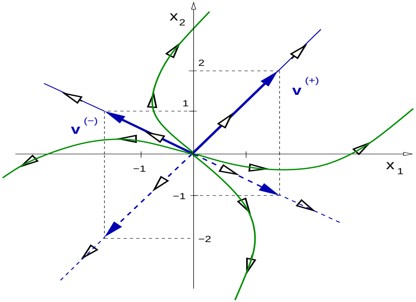

Exam: November 12, 2008. Variation of Problem 4.

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Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)} e^{\lambda_- t}$,

Solution:
Here $\lambda_+ > 0 > \lambda_-$. We plot the solutions

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Exam: November 12, 2008. Variation of Problem 4.

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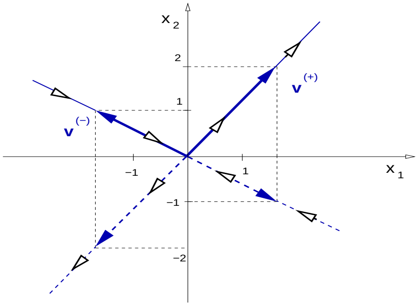

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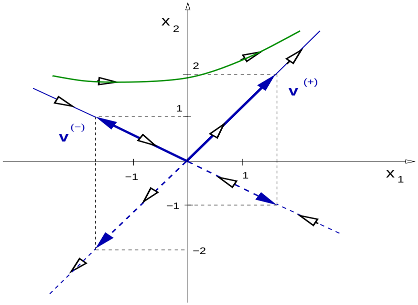

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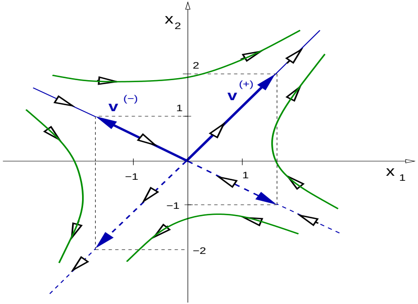

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Extra problem.

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Solution: Eigenvalues of A:

Extra problem.

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Extra problem.

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Extra problem.

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Hence $\lambda_+ = \lambda_- = -1$.

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$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

Extra problem.

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Extra problem.

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$v_1 = 2\,v_2.$

Extra problem.

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$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

$v_1 = 2\,v_2$. Choosing $v_1 = 2$

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

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$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

$v_1 = 2\,v_2$. Choosing $v_1 = 2$ and $v_2 = 1$,

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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$v_1 = 2\,v_2$. Choosing $v_1 = 2$ and $v_2 = 1$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_\pm = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Extra problem.

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$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right]$$

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Find $\mathbf{x}$ solution of the IVP

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Hence $w_1 = 2w_2 - 1$,

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \;\Rightarrow\; \left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Choose $w_2 = 0$, so $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Extra problem.

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Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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Fundamental sol: $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$,

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Fundamental sol: $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

Extra problem.

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General sol: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

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The solution is $\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 5 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}.$ $\triangleleft$

Extra problem.

Example

Let $\lambda = -1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm\mathbf{x}^{(1)} = \pm\mathbf{v}\, e^{-t}$ and $\pm\mathbf{x}^{(2)} = \pm\big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{-t}$.

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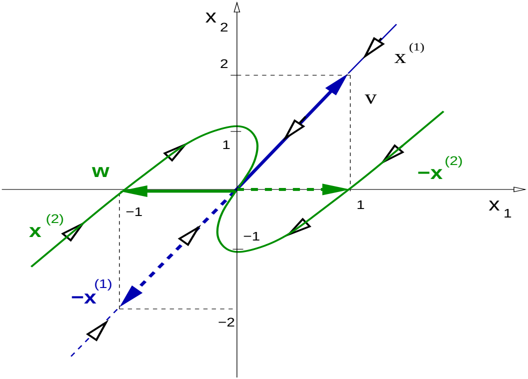

Extra problem.

Example

Let $\lambda = 1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm\mathbf{x}^{(1)} = \pm\mathbf{v}\, e^{t}$ and $\pm\mathbf{x}^{(2)} = \pm\big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{t}$.

Extra problem.

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Let $\lambda = 1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

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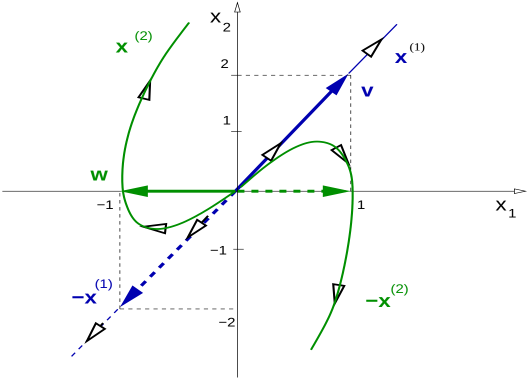

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Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\,e^{\alpha t},\ \mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\,e^{\alpha t}.$$

for the cases $\alpha = 0$, and $\alpha > 0$, where $\beta > 0$.

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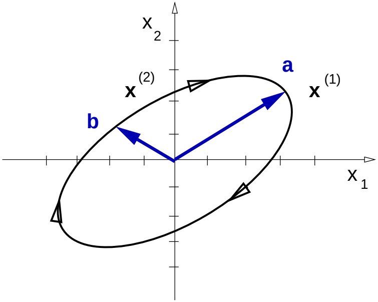

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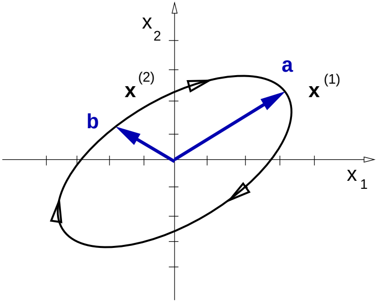

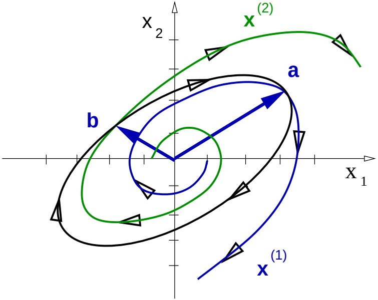

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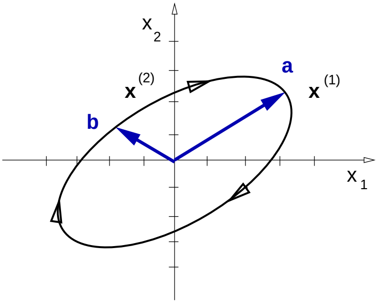

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