Complex, distinct eigenvalues (Sect. 7.6)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- Real matrix with a pair of complex eigenvalues.
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Remark:
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(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 7.8).

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Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

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Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 7.8).

Complex, distinct eigenvalues (Sect. 7.6)

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Review: The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Complex, distinct eigenvalues (Sect. 7.6)

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Real matrix with a pair of complex eigenvalues.

Theorem

If $\{\lambda, \mathbf{v}\}$ *is an eigen-pair of an* $n \times n$ *real-valued matrix* A*, then* $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ *also is an eigen-pair of matrix* A*.*

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If $\{\lambda, \mathbf{v}\}$ is an eigen-pair of an $n \times n$ real-valued matrix A, then $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ also is an eigen-pair of matrix A.

Proof: By hypothesis $A\,\mathbf{v} = \lambda\,\mathbf{v}$ and $\overline{A} = A$.

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Therefore $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ is an eigen-pair of matrix A. □

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Remark: The Theorem above is equivalent to the following:

Real matrix with a pair of complex eigenvalues.

Theorem

If $\{\lambda, \mathbf{v}\}$ is an eigen-pair of an $n \times n$ real-valued matrix A, then $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ also is an eigen-pair of matrix A.

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$$\overline{A\,\mathbf{v}} = \overline{\lambda\,\mathbf{v}} \quad \Leftrightarrow \quad \overline{A}\,\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}} \quad \Leftrightarrow \quad A\,\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}.$$

Therefore $\{\overline{\lambda}, \overline{\mathbf{v}}\}$ is an eigen-pair of matrix A. □

Remark: The Theorem above is equivalent to the following: If an $n \times n$ real-valued matrix A has eigen pairs

$$\lambda_1 = \alpha + i\beta, \quad \mathbf{v}_1 = \mathbf{a} + i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then so is

$$\lambda_2 = \alpha - i\beta, \quad \mathbf{v}_2 = \mathbf{a} - i\mathbf{b}.$$

Real matrix with a pair of complex eigenvalues.

Theorem (Complex pairs)

If an $n \times n$ real-valued matrix A has eigen pairs

$$\lambda_{\pm} = \alpha \pm i\beta, \quad \mathbf{v}^{(\pm)} = \mathbf{a} \pm i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then the differential equation

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

has a linearly independent set of two complex-valued solutions

$$\mathbf{x}^{(+)} = \mathbf{v}^{(+)}\, e^{\lambda_+ t}, \qquad \mathbf{x}^{(-)} = \mathbf{v}^{(-)}\, e^{\lambda_- t},$$

and it also has a linearly independent set of two real-valued solutions

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\, \cos(\beta t) - \mathbf{b}\, \sin(\beta t)\big]\, e^{\alpha t},$$

$$\mathbf{x}^{(2)} = \big[\mathbf{a}\, \sin(\beta t) + \mathbf{b}\, \cos(\beta t)\big]\, e^{\alpha t}.$$

Real matrix with a pair of complex eigenvalues.

Proof: We know that one solution to the differential equation is

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$$\mathbf{x}^{(+)} = \mathbf{v}^{(+)}\, e^{\lambda_+ t} = (\mathbf{a} + i\mathbf{b})\, e^{(\alpha + i\beta)t}$$

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Euler equation implies

$$\mathbf{x}^{(+)} = (\mathbf{a} + i\mathbf{b}) e^{\alpha t} \left[\cos(\beta t) + i \sin(\beta t)\right],$$

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A similar calculation done on $\mathbf{x}^{(-)}$ implies

$$\mathbf{x}^{(-)} = \left[\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t)\right] e^{\alpha t} - i \left[\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t)\right] e^{\alpha t}.$$

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Introduce $\mathbf{x}^{(1)} = (\mathbf{x}^{(+)} + \mathbf{x}^{(-)})/2$,

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□

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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$$A - \lambda_+ I = A - (2+3i)I = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}.$$

Real matrix with a pair of complex eigenvalues.

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$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

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$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: $\lambda_{\pm} = 2 \pm 3i$, $(A - \lambda_+ I) = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix}$.

We need to solve $(A - \lambda_+ I)\,\mathbf{v}^{(+)} = \mathbf{0}$ for $\mathbf{v}^{(+)}$. Gauss operations

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix}$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

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So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$. Choose

$$v_2 = 1, \quad v_1 = -i,$$

Real matrix with a pair of complex eigenvalues.

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So, the eigenvector $\mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $v_1 = -iv_2$. Choose

$$v_2 = 1, \quad v_1 = -i, \quad \Rightarrow \quad \mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_+ = 2 + 3i.$$

Real matrix with a pair of complex eigenvalues.

Example

Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

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Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$,

Real matrix with a pair of complex eigenvalues.

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Notice that $\mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$.

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The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$

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The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i$ implies

$$\alpha = 2,$$

Real matrix with a pair of complex eigenvalues.

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Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: Recall: eigenvalues $\lambda_{\pm} = 2 \pm 3i$, and $\mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

The second eigenvector is $\mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)}$, that is, $\mathbf{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

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The notation $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$ implies

$$\alpha = 2, \qquad \beta = 3,$$

Real matrix with a pair of complex eigenvalues.

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Find a real-valued set of fundamental solutions to the equation

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

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Real matrix with a pair of complex eigenvalues.

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Real-valued solutions are $\mathbf{x}^{(1)} = \big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big]\, e^{\alpha t}$, and

Real matrix with a pair of complex eigenvalues.

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Real matrix with a pair of complex eigenvalues.

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$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t}$$

Real matrix with a pair of complex eigenvalues.

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$$\mathbf{x}^{(1)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}.$$

Real matrix with a pair of complex eigenvalues.

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Find a real-valued set of fundamental solutions to the equation

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Real matrix with a pair of complex eigenvalues.

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$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos(3t)\right) e^{2t} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}.$$

$\triangleleft$

Complex, distinct eigenvalues (Sect. 7.6)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- Real matrix with a pair of complex eigenvalues.
- **Phase portraits for 2×2 systems.**

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

The phase portrait of the vectors

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

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is a radius one circle.

Phase portraits for 2 × 2 systems.

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Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

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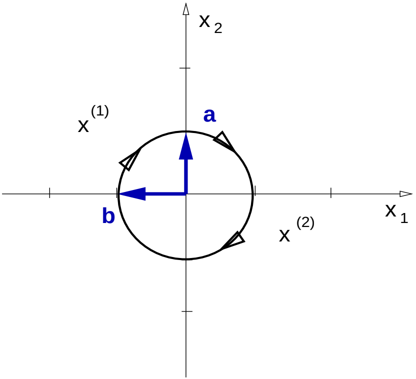

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:

The phase portrait of the solutions

$$\tilde{\mathbf{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

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Phase portraits for 2×2 systems.

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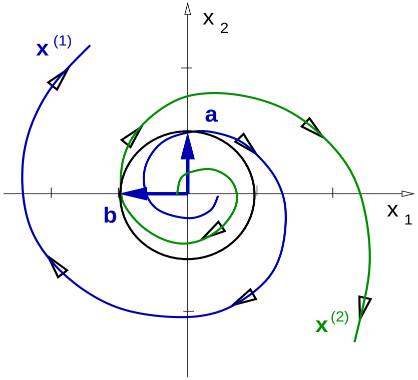

Phase portraits for 2×2 systems.

Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\,e^{\alpha t},\ \mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\,e^{\alpha t}.$$

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Phase portraits for 2×2 systems.

Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \left[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\right] e^{\alpha t}, \; \mathbf{x}^{(2)} = \left[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\right] e^{\alpha t}.$$

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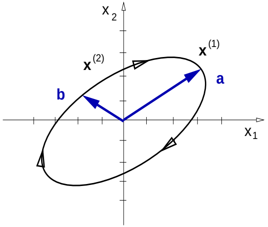

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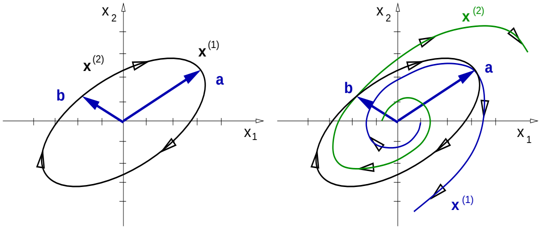

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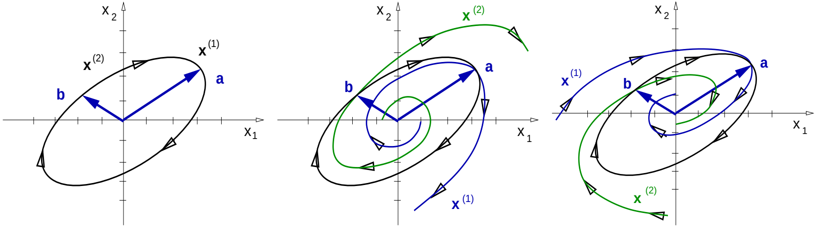

Complex, distinct eigenvalues (Sect. 7.8)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- The algebraic multiplicity of an eigenvalue.
- Non-diagonalizable matrices with a repeated eigenvalue.
- Phase portraits for 2×2 systems.

Review: Classification of 2×2 diagonalizable systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

Review: Classification of 2×2 diagonalizable systems.

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Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 7.5).

Review: Classification of 2×2 diagonalizable systems.

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(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 7.6).

Review: Classification of 2×2 diagonalizable systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 7.5).

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(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 7.8).

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Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

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(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 7.8).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 7.8).

Complex, distinct eigenvalues (Sect. 7.8)

- Review: Classification of 2×2 diagonalizable systems.
- **Review: The case of diagonalizable matrices.**
- The algebraic multiplicity of an eigenvalue.
- Non-diagonalizable matrices with a repeated eigenvalue.
- Phase portraits for 2×2 systems.

Review: The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Complex, distinct eigenvalues (Sect. 7.8)

- Review: Classification of 2×2 diagonalizable systems.
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The algebraic multiplicity of an eigenvalue.

Definition

Let $\{\lambda_1, \cdots, \lambda_k\}$ be the set of eigenvalues of an $n \times n$ matrix, where $1 \leqslant k \leqslant n$, hence the characteristic polynomial is

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k}.$$

The positive integer r_i, for $i = 1, \cdots, k$, is called the *algebraic multiplicity* of the eigenvalue λ_i. The eigenvalue λ_i is called *repeated* iff $r_i > 1$.

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Remark:

- A matrix with repeated eigenvalues may or may not be diagonalizable.

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Remark:

- A matrix with repeated eigenvalues may or may not be diagonalizable.
- Equivalently: An $n \times n$ matrix with repeated eigenvalues may or may not have a linearly independent set of n eigenvectors.

The algebraic multiplicity of an eigenvalue.

Example

Show that matrix A is diagonalizable but matrix B is not, where

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Solution: The eigenvalues of A are the solutions of

$$\begin{vmatrix} (3-\lambda) & 0 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix}$$

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Verify that the eigenvalues are: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$.

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We conclude: A is diagonalizable.

The algebraic multiplicity of an eigenvalue.

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Solution: The eigenvalues of B are the solutions of

The algebraic multiplicity of an eigenvalue.

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The algebraic multiplicity of an eigenvalue.

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Solution: The eigenvalues of B are the solutions of

$$\begin{vmatrix} (3-\lambda) & 1 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda - 3)^2 (\lambda - 1) = 0,$$

We conclude: $\lambda_1 = 3$, $r_1 = 2$, and $\lambda_2 = 1$, $r_2 = 1$.

Verify that the eigenvalues are: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

We conclude: B is not diagonalizable.

The algebraic multiplicity of an eigenvalue.

Example

Find a fundamental set of solutions to

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

Solution: Since matrix A is diagonalizable, with eigen-pairs,

$$\lambda_1 = 3, \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad \lambda_2 = 1, \quad \left\{ \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

The algebraic multiplicity of an eigenvalue.

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We conclude that a set of fundamental solutions is

$$\left\{ \mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t},\ \mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{3t},\ \mathbf{x}_3(t) = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} e^{t} \right\}. \quad \triangleleft$$

Complex, distinct eigenvalues (Sect. 7.8)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- The algebraic multiplicity of an eigenvalue.
- **Non-diagonalizable matrices with a repeated eigenvalue.**
- Phase portraits for 2×2 systems.

Non-diagonalizable matrices with a repeated eigenvalue.

Theorem (Repeated eigenvalue)

If λ *is an eigenvalue of an* $n \times n$ *matrix* A *having algebraic multiplicity* $r = 2$ *and only one associated eigen-direction, then the differential equation*

$$\mathbf{x}'(t) = A\,\mathbf{x}(t),$$

has a linearly independent set of solutions given by

$$\left\{\mathbf{x}^{(1)}(t) = \mathbf{v}\ e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \left(\mathbf{v}\ t + \mathbf{w}\right) e^{\lambda t}\right\}.$$

where the vector $\mathbf{w}$ *is solution of*

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

which always has a solution $\mathbf{w}$.

Non-diagonalizable matrices with a repeated eigenvalue.

Recall: The case of a single second order equation

$$y'' + a_1\, y' + a_0\, y = 0$$

Non-diagonalizable matrices with a repeated eigenvalue.

Recall: The case of a single second order equation

$$y'' + a_1\, y' + a_0\, y = 0$$

with characteristic polynomial

$$p(r) = r^2 + a_1\, r + a_0$$

Non-diagonalizable matrices with a repeated eigenvalue.

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In this case a fundamental set of solutions is

$$\big\{ y_1(t) = e^{r_1 t}, \quad y_2(t) = t\, e^{r_1 t} \big\}.$$

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$$\big\{ y_1(t) = e^{r_1 t}, \quad y_2(t) = t\, e^{r_1 t} \big\}.$$

This is not the case with systems of first order linear equations,

$$\big\{ \mathbf{x}^{(1)}(t) = \mathbf{v}\; e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\; t + \mathbf{w} \big)\, e^{\lambda t} \big\}.$$

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In general, $\mathbf{w} \neq \mathbf{0}$.

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \dfrac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \dfrac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A.

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Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A. Its characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(-\frac{3}{2} - \lambda\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} - \lambda\right) \end{vmatrix}$$

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So $p(\lambda) = \lambda^2 + 2\lambda + 1$

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So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$.

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So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. The roots and multiplicity are

$$\lambda = -1, \qquad r = 2.$$

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The corresponding eigenvectors are the solutions of $(A + I)\mathbf{v} = \mathbf{0}$,

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Solution: Recall: $\lambda = -1$, with $r = 2$, and $(A + I) \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

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The eigenvector components satisfy: $v_1 = 2v_2$.

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The eigenvector components satisfy: $v_1 = 2v_2$. We obtain,

$$\lambda = -1, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2.$$

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Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Recall: $\lambda = -1$, with $r = 2$, and $(A + I) \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

The eigenvector components satisfy: $v_1 = 2v_2$. We obtain,

$$\lambda = -1, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2.$$

We conclude that this eigenvalue has only one eigen-direction.

Non-diagonalizable matrices with a repeated eigenvalue.

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$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right]$$

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We obtain $w_1 = 2w_2 - 4$.

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Non-diagonalizable matrices with a repeated eigenvalue.

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Given a solution $\mathbf{w}$, then $c\mathbf{v} + \mathbf{w}$ is also a solution, $c \in \mathbb{R}$.

Non-diagonalizable matrices with a repeated eigenvalue.

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We choose the simplest solution, $\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

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We obtain $w_1 = 2w_2 - 4$. That is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

Given a solution $\mathbf{w}$, then $c\mathbf{v} + \mathbf{w}$ is also a solution, $c \in \mathbb{R}$.

We choose the simplest solution, $\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$. We conclude,

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) e^{-t}. \qquad \triangleleft$$

Non-diagonalizable matrices with a repeated eigenvalue.

Example

Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4} \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Non-diagonalizable matrices with a repeated eigenvalue.

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$$\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

Non-diagonalizable matrices with a repeated eigenvalue.

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Find the solution $\mathbf{x}$ to the IVP

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The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Non-diagonalizable matrices with a repeated eigenvalue.

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Non-diagonalizable matrices with a repeated eigenvalue.

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Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Non-diagonalizable matrices with a repeated eigenvalue.

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Find the solution $\mathbf{x}$ to the IVP

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$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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We conclude: $\mathbf{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{4} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$ $\triangleleft$

Complex, distinct eigenvalues (Sect. 7.8)

- Review: Classification of 2×2 diagonalizable systems.
- Review: The case of diagonalizable matrices.
- The algebraic multiplicity of an eigenvalue.
- Non-diagonalizable matrices with a repeated eigenvalue.
- **Phase portraits for 2×2 systems.**

Phase portraits for 2×2 systems.

Example

Sketch a phase portrait for solutions of

$$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

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We start plotting the vectors

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

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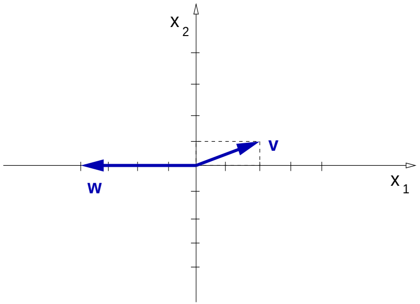

Phase portraits for 2×2 systems.

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$$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4} \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution:

Now plot the solutions

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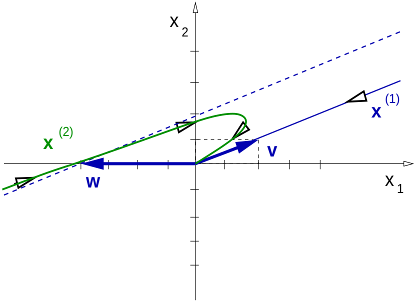

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Solution:

Now plot the solutions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

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This is the case $\lambda < 0$.

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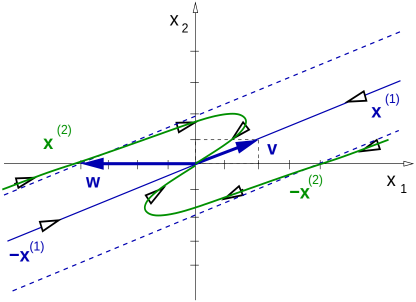

Phase portraits for 2×2 systems.

Example

Given any vectors $\mathbf{v}$ and $\mathbf{w}$, and any constant λ, plot the phase portraits of the functions

$$\mathbf{x}^{(1)}(t) = \mathbf{v}\, e^{\lambda t}, \qquad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{\lambda t},$$

Solution:

The case $\lambda < 0$. We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

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Phase portraits for 2×2 systems.

Example

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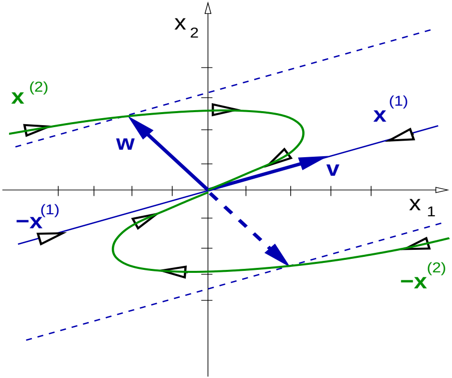

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The case $\lambda > 0$. We plot the functions

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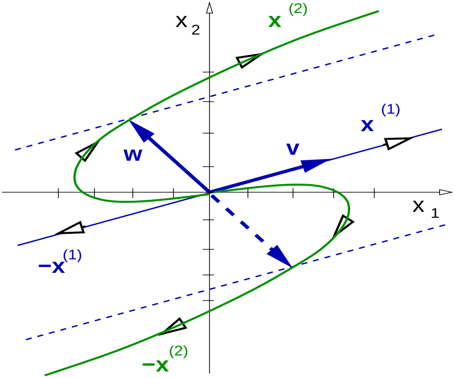

Review of Chapter 7.

- Review of Sections 7.5, 7.6, 7.8.
- Const. Coeff., homogeneours linear differential systems:
  - Real, different eigenvalues (7.5).
  - Complex, different eigenvalues (7.6).
  - Repeated eigenvalues (7.8).

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

Exam: November 12, 2008. Problem 4.

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Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix}$$

Exam: November 12, 2008. Problem 4.

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Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

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Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0$$

Exam: November 12, 2008. Problem 4.

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$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}\big[-5 \pm \sqrt{25-16}\big]$$

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}\big[-5 \pm \sqrt{25-16}\big] = \frac{1}{2}\big[-5 \pm 3\big]$$

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

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Hence $\lambda_+ = -1$,

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Solution: Eigenvalues of A:

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Hence $\lambda_+ = -1$, $\lambda_- = -4$.

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Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

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Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for λ_+.

$$(A+I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}$$

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Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

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$$(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix}$$

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Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

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Exam: November 12, 2008. Problem 4.

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$2v_1 = \sqrt{2}\,v_2.$

Exam: November 12, 2008. Problem 4.

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$2v_1 = \sqrt{2}\,v_2$. Choosing $v_1 = \sqrt{2}$

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Exam: November 12, 2008. Problem 4.

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Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

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$$(A+I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$2v_1 = \sqrt{2}\,v_2$. Choosing $v_1 = \sqrt{2}$ and $v_2 = 2$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

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Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Exam: November 12, 2008. Problem 4.

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Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Exam: November 12, 2008. Problem 4.

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Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

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Eigenvector for λ_-.

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

Exam: November 12, 2008. Problem 4.

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$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix}$$

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$v_1 = -\sqrt{2}\, v_2.$

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$v_1 = -\sqrt{2}\,v_2$. Choosing $v_1 = -\sqrt{2}$

Exam: November 12, 2008. Problem 4.

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$v_1 = -\sqrt{2}\,v_2$. Choosing $v_1 = -\sqrt{2}$ and $v_2 = 1$,

Exam: November 12, 2008. Problem 4.

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Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$,

Exam: November 12, 2008. Problem 4.

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Exam: November 12, 2008. Problem 4.

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Eigenvector for λ_-.

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$v_1 = -\sqrt{2}\,v_2$. Choosing $v_1 = -\sqrt{2}$ and $v_2 = 1$, so, $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$.

General solution: $\mathbf{x} = c_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$. ◁

Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

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Solution:

We start plotting the vectors

$$\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix},$$

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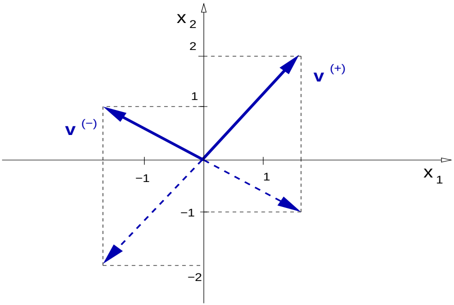

Exam: November 12, 2008. Problem 4.

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Solution:

We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

$$\mathbf{x}^{(-)}, \quad -\mathbf{x}^{(-)}.$$

Exam: November 12, 2008. Problem 4.

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Plot the phase portrait of several linear combinations of the fundamental solutions found above,

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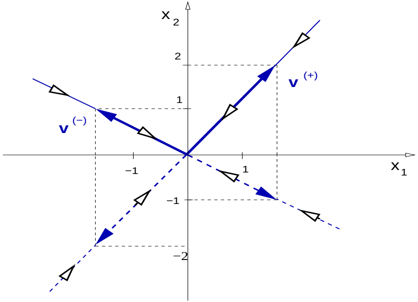

Exam: November 12, 2008. Problem 4.

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Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

Solution:

Recall: $\lambda_- < \lambda_+ < 0$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)}\, e^{-t} + \mathbf{v}^{(-)}\, e^{-4t}.$$

Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

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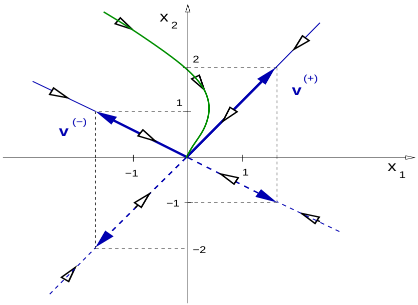

Exam: November 12, 2008. Problem 4.

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$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

Solution:

We plot the solutions

$$\mathbf{x} = c_1\,\mathbf{x}^{(+)} + c_2\,\mathbf{x}^{(-)},$$

for $c_1 = \pm 1$ and $c_2 = \pm 1$.

Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

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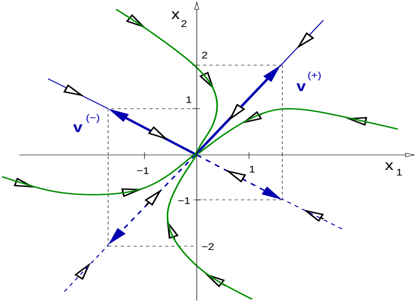

Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)} e^{\lambda_- t}$,

Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

Solution:

Here $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

$$\mathbf{x}^{(-)}, \quad -\mathbf{x}^{(-)}.$$

Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.
Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

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Here $\lambda_+ > \lambda_- > 0$. We plot the solutions

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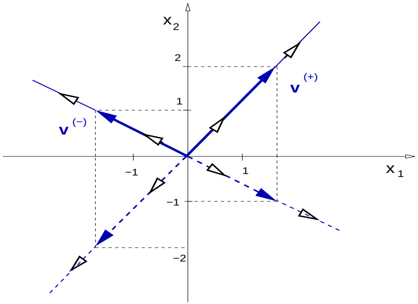

Exam: November 12, 2008. Variation of Problem 4.

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Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)} e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)} e^{\lambda_- t}$,

Solution:

Recall: $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)} e^{4t} + \mathbf{v}^{(-)} e^{t}.$$

Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

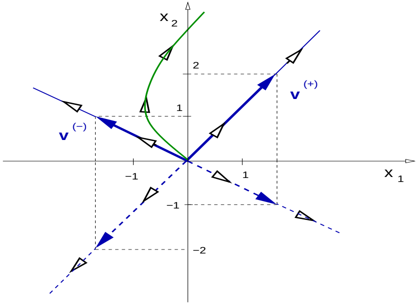
Solution:

Recall: $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)}\, e^{4t} + \mathbf{v}^{(-)}\, e^{t}.$$



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$$\mathbf{x} = c_1 \, \mathbf{x}^{(+)} + c_2 \, \mathbf{x}^{(-)},$$

for $c_1 = \pm 1$ and $c_2 = \pm 1$.

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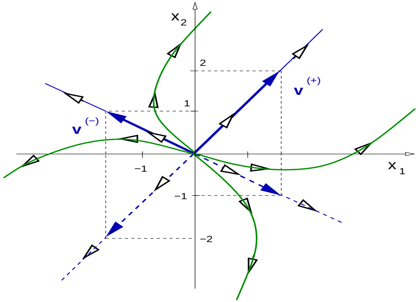

Exam: November 12, 2008. Variation of Problem 4.

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Solution:
Here $\lambda_+ > 0 > \lambda_-$. We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

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Exam: November 12, 2008. Variation of Problem 4.

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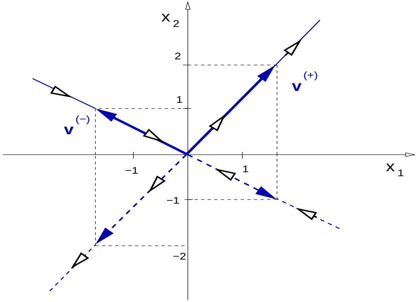

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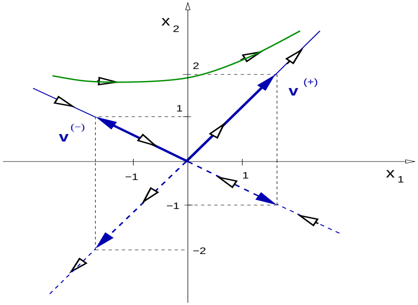

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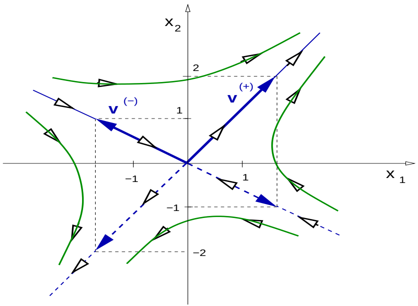

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Extra problem.

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Solution: Eigenvalues of A:

Extra problem.

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Extra problem.

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Extra problem.

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Hence $\lambda_+ = \lambda_- = -1$.

Extra problem.

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$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

Extra problem.

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Extra problem.

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$v_1 = 2\,v_2.$

Extra problem.

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$v_1 = 2\,v_2$. Choosing $v_1 = 2$

Extra problem.

Example

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$v_1 = 2\,v_2$. Choosing $v_1 = 2$ and $v_2 = 1$,

Extra problem.

Example

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$v_1 = 2\,v_2$. Choosing $v_1 = 2$ and $v_2 = 1$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Extra problem.

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Find $\mathbf{x}$ solution of the IVP

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Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right]$$

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Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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Hence $w_1 = 2w_2 - 1$,

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

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Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Extra problem.

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Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

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Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Choose $w_2 = 0$, so $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Extra problem.

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Fundamental sol: $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$,

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Fundamental sol: $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

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$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Fundamental sol: $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

General sol: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}.$

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Initial condition: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix},$

Extra problem.

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Extra problem.

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Solution: Recall: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}.$

Initial condition: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix},$

that is, $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, also, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$

The solution is $\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 5 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}.$ $\lhd$

Extra problem.

Example

Let $\lambda = -1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm \mathbf{x}^{(1)} = \pm \mathbf{v}\, e^{-t}$ and $\pm \mathbf{x}^{(2)} = \pm \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{-t}$.

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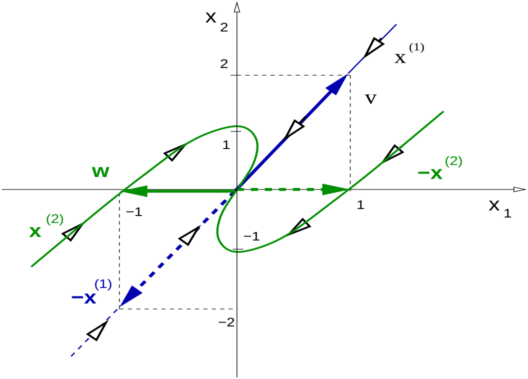
Solution:



Extra problem.

Example

Let $\lambda = 1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm\mathbf{x}^{(1)} = \pm\mathbf{v}\, e^{t}$ and $\pm\mathbf{x}^{(2)} = \pm\big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{t}$.

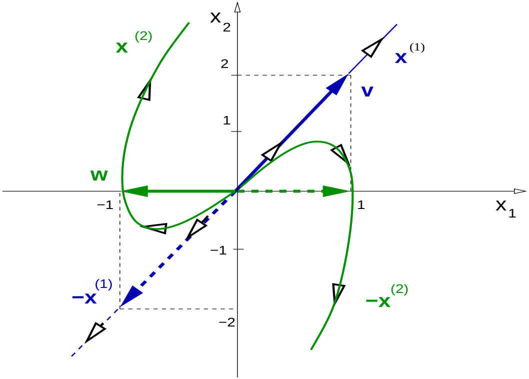
Extra problem.

Example

Let $\lambda = 1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm \mathbf{x}^{(1)} = \pm \mathbf{v}\, e^t$ and $\pm \mathbf{x}^{(2)} = \pm \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^t$.

Solution:



Extra problem.

Example

Given any vectors **a** and **b**, sketch qualitative phase portraits of

$$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\,e^{\alpha t},\ \mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\,e^{\alpha t}.$$

for the cases $\alpha = 0$, and $\alpha > 0$, where $\beta > 0$.

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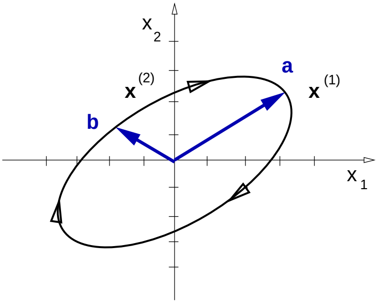

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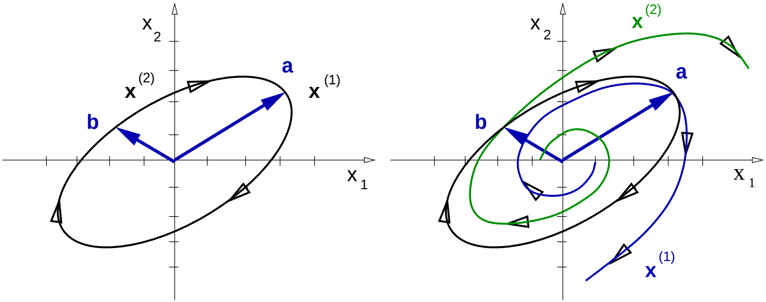

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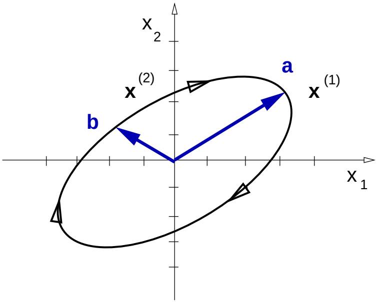

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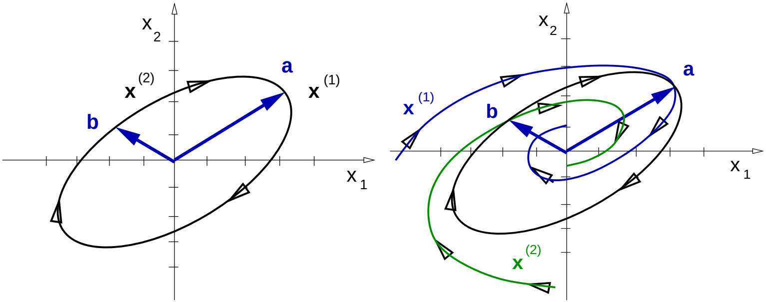

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