Review of Linear Algebra (Sect. 7.3)

- Eigenvalues, eigenvectors of a matrix.
- Computing eigenvalues and eigenvectors.
- Diagonalizable matrices.
- The case of Hermitian matrices.

Eigenvalues, eigenvectors of a matrix

Definition

A number λ and a non-zero n-vector $\mathbf{v}$ are respectively called an *eigenvalue* and *eigenvector* of an $n \times n$ matrix A iff the following equation holds,

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Example

Verify that the pair $\lambda_1 = 4$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvalue and eigenvector pairs of matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Solution: $A\mathbf{v}_1$

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Eigenvalues, eigenvectors of a matrix

Remarks:

- If we interpret an $n \times n$ matrix A as a function $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then the eigenvector $\mathbf{v}$ determines a particular *direction* on $\mathbb{R}^n$ where the action of A is *simple*:

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Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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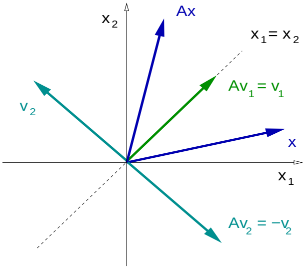

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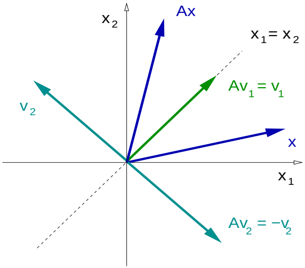

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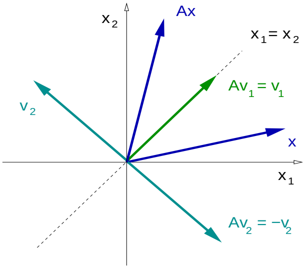

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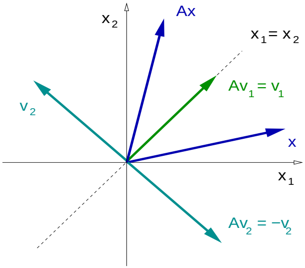

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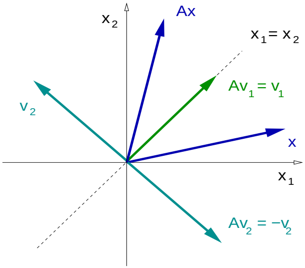

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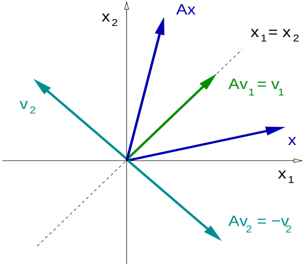

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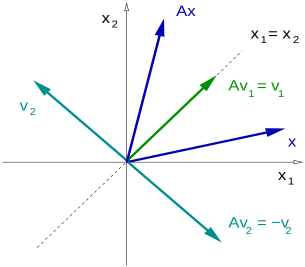

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An eigenvalue eigenvector pair is: $\lambda_1 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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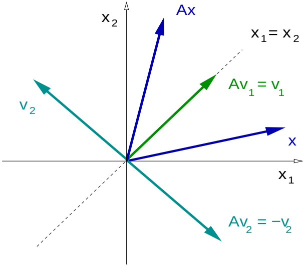

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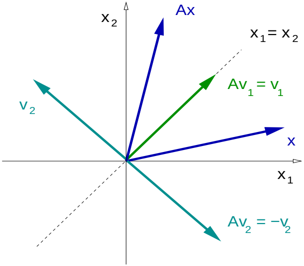

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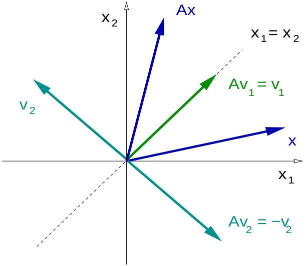

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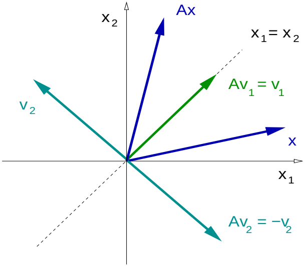

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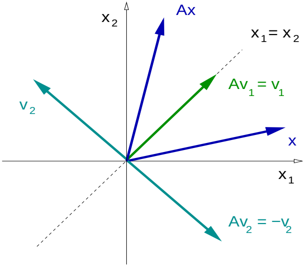

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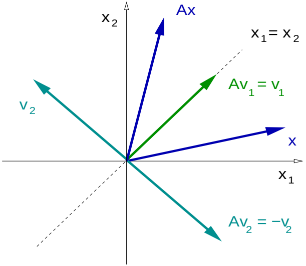

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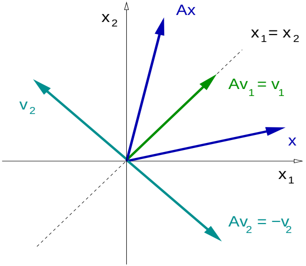
Eigenvalues, eigenvectors of a matrix

Example

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



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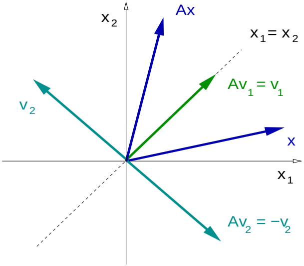

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Fix $\theta \in (0, \pi)$ and define $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

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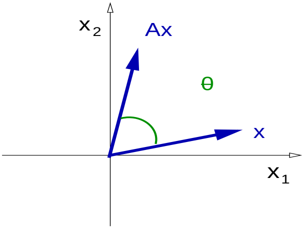

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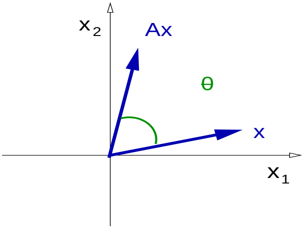

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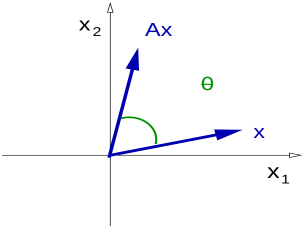

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Remark:

Matrix A has complex-values eigenvalues and eigenvectors.

Review of Linear Algebra (Sect. 7.3)

- Eigenvalues, eigenvectors of a matrix.
- **Computing eigenvalues and eigenvectors.**
- Diagonalizable matrices.
- The case of Hermitian matrices.

Computing eigenvalues and eigenvectors.

Problem:

Given an $n \times n$ matrix A, find, if possible, λ and $\mathbf{v} \neq \mathbf{0}$ solution of

$$A\mathbf{v} = \lambda\,\mathbf{v}.$$

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(a) First solve for λ.

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Solution:

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Computing eigenvalues and eigenvectors.

Theorem (Eigenvalues-eigenvectors)

(a) *The number* λ *is an eigenvalue of an* $n \times n$ *matrix* A *iff*

$$\det(A - \lambda I) = 0.$$

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Find λ such that for a non-zero vector $\mathbf{v}$ holds,

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Since $(A - \lambda I)$ is not invertible, then $\det(A - \lambda I) = 0$.

Once λ is known, the original eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$. $\square$

Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Computing eigenvalues and eigenvectors.

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Computing eigenvalues and eigenvectors.

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Compute the eigenvector for $\lambda_+ = 4$. Solve $(A - 4I)\mathbf{v}_+ = \mathbf{0}$.

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Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

$$\mathbf{v}_+ = \begin{bmatrix} v_2^+ \\ v_2^+ \end{bmatrix}$$

Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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The first eigenvalue eigenvector pair is $\lambda_+ = 4$, $\mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Computing eigenvalues and eigenvectors.

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Computing eigenvalues and eigenvectors.

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The second eigenvalue eigenvector pair: $\lambda_- = -2$, $\mathbf{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\triangleleft$

Review of Linear Algebra (Sect. 7.3)

- Eigenvalues, eigenvectors of a matrix.
- Computing eigenvalues and eigenvectors.
- **Diagonalizable matrices.**
- The case of Hermitian matrices.

Diagonalizable matrices.

Definition

An $n \times n$ matrix D is called *diagonal* iff $D = \begin{bmatrix} d_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{nn} \end{bmatrix}$.

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An $n \times n$ matrix A is called *diagonalizable* iff there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

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- Systems of linear *differential* equations are simple to solve in the case that the coefficient matrix A is diagonalizable.

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- Systems of linear *differential* equations are simple to solve in the case that the coefficient matrix A is diagonalizable.
- In such case, it is simple to *decouple* the differential equations.
- One solves the decoupled equations, and then transforms back to the original unknowns.

Diagonalizable matrices.

Theorem (Diagonalizability and eigenvectors)

An $n \times n$ matrix A is diagonalizable iff matrix A has a linearly independent set of n eigenvectors. Furthermore,

$$A = PDP^{-1}, \quad P = [\mathbf{v}_1, \cdots, \mathbf{v}_n], \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\lambda_i, \mathbf{v}_i$, for $i = 1, \cdots, n$, are eigenvalue-eigenvector pairs of A.

Diagonalizable matrices.

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Remark: It is not simple to know whether an $n \times n$ matrix A has a linearly independent set of n eigenvectors.

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Diagonalizable matrices.

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Remark: It is not simple to know whether an $n \times n$ matrix A has a linearly independent set of n eigenvectors. One simple case is given in the following result.

Theorem (n different eigenvalues)

If an $n \times n$ matrix A has n different eigenvalues, then A is diagonalizable.

Diagonalizable matrices.

Example

Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Diagonalizable matrices.

Example

Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: We known that the eigenvalue eigenvector pairs are

$$\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Diagonalizable matrices.

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Introduce P and D as follows,

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Introduce P and D as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

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Then

$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

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$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Diagonalizable matrices.

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Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

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$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

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We conclude,

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that is, A is diagonalizable. $\triangleleft$

Review of Linear Algebra (Sect. 7.3)

- Eigenvalues, eigenvectors of a matrix.
- Computing eigenvalues and eigenvectors.
- Diagonalizable matrices.
- **The case of Hermitian matrices.**

The case of Hermitian matrices.

Definition

An $n \times n$ matrix A is called Hermitian iff $A = A^*$.
An $n \times n$ matrix A is called symmetric iff $A = A^T$.

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Every Hermitian matrix is diagonalizable.

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Every Hermitian matrix is diagonalizable.

Remark: A real-valued Hermitian matrix A is symmetric, since

$$A = A^* = \overline{A}^T = A^T \quad \Rightarrow \quad A = A^T$$

Example

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 7 \\ 3 & 7 & 11 \end{bmatrix}$ is symmetric,

The case of Hermitian matrices.

Definition

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Example

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 7 \\ 3 & 7 & 11 \end{bmatrix}$ is symmetric, $B = \begin{bmatrix} 1 & -i & 1 \\ i & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ is Hermitian.

Properties of differential linear systems (Sect. 7.4)

- Review: $n \times n$ linear differential systems.
- Fundamental solutions to homogeneous systems.
- Existence and uniqueness of solutions to IVP.
- The Wronskian of n solutions.

Review: $n \times n$ linear differential systems.

Definition

An *$n \times n$ linear differential system* is a the following: Given an $n \times n$ matrix-valued function A, and an n-vector-valued function $\mathbf{b}$, find an n-vector-valued function $\mathbf{x}$ solution of

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$

The system above is called *homogeneous* iff holds $\mathbf{b} = 0$.

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$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix},$$

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$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t) \;\Leftrightarrow\; \begin{aligned} x_1' &= a_{11}(t)\,x_1 + \cdots + a_{1n}(t)\,x_n + b_1(t) \\ &\vdots \\ x_n' &= a_{n1}(t)\,x_1 + \cdots + a_{nn}(t)\,x_n + b_n(t). \end{aligned}$$

Review: $n \times n$ linear differential systems.

Example

Find the explicit expression for the linear system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ in the case that

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} e^{t} \\ 2e^{3t} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

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Solution: The 2×2 linear system is given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}.$$

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That is,

$$x_1'(t) = x_1(t) + 3x_2(t) + e^t,$$
$$x_2'(t) = 3x_1(t) + x_2(t) + 2e^{3t}.$$

◁

Review: $n \times n$ linear differential systems.

Remark: Derivatives of vector-valued functions are computed component-wise.

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Definition

An $n \times n$ matrix-valued function with values $A(t) = [a_{ij}(t)]$ is called *continuous* iff every coefficient a_{ij} is a continuous function.

Properties of differential linear systems (Sect. 7.4)

- Review: $n \times n$ linear differential systems.
- **Fundamental solutions to homogeneous systems.**
- Existence and uniqueness of solutions to IVP.
- The Wronskian of n solutions.

Fundamental solutions to homogeneous systems.

Definition

A linearly independent set of solutions $\{\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)\}$ of the $n \times n$ homogeneous linear differential system

$$\mathbf{x}' = A(t)\,\mathbf{x} \tag{1}$$

is called a *fundamental set* of solutions, and the function

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t),$$

is called the *general solution* of Eq. (1), where $c_1, \cdots, c_n$ are arbitrary constants. The $n \times n$ matrix-valued function

$$X(t) = \left[\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)\right]$$

is called a *fundamental matrix* of the Eq. (1), and the function

$$w(t) = \det\big(X(t)\big)$$

is called the *Wronskian* of the fundamental solutions.

Fundamental solutions to homogeneous systems.

Example

Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\, e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\, e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

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Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\, e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\, e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

Solution: First we verify the $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions.

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$$\mathbf{x}^{(1)\prime}(t) = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix}'$$

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Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\,e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

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$$A\,\mathbf{x}^{(1)}(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

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$$A\,\mathbf{x}^{(1)}(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} 6-2 \\ 4-2 \end{bmatrix} e^{2t}$$

Fundamental solutions to homogeneous systems.

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Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\,e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

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We conclude; $\mathbf{x}^{(1)\prime}(t) = A\,\mathbf{x}^{(1)}(t)$.

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Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\,e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

Solution: Recall: $\mathbf{x}^{(1)\prime}(t) = A\,\mathbf{x}^{(1)}(t)$.

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Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\, e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\, e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

Solution: Recall: $\mathbf{x}^{(1)\prime}(t) = A\,\mathbf{x}^{(1)}(t)$.

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Solution: Recall: $\mathbf{x}^{(1)\prime}(t) = A\,\mathbf{x}^{(1)}(t)$.

$$\mathbf{x}^{(2)\prime}(t) = \begin{bmatrix} e^{-t} \\ 2\,e^{2t} \end{bmatrix}' = \begin{bmatrix} -e^{-t} \\ -2\,e^{-t} \end{bmatrix}$$

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$$\mathbf{x}^{(2)\prime}(t) = \begin{bmatrix} e^{-t} \\ 2\,e^{2t} \end{bmatrix}' = \begin{bmatrix} -e^{-t} \\ -2\,e^{-t} \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}^{(2)\prime}(t) = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}.$$

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$$A\,\mathbf{x}^{(2)}(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} = \begin{bmatrix} 3-4 \\ 2-4 \end{bmatrix} e^{-t}$$

Fundamental solutions to homogeneous systems.

Example

Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\,e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

Solution: Recall: $\mathbf{x}^{(1)\prime}(t) = A\,\mathbf{x}^{(1)}(t)$.

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$$A\,\mathbf{x}^{(2)}(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} = \begin{bmatrix} 3-4 \\ 2-4 \end{bmatrix} e^{-t} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

Fundamental solutions to homogeneous systems.

Example

Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\,e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

Solution: Recall: $\mathbf{x}^{(1)\prime}(t) = A\,\mathbf{x}^{(1)}(t)$.

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We conclude; $\mathbf{x}^{(2)\prime}(t) = A\,\mathbf{x}^{(2)}(t)$.

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We need to compute the determinant of the fundamental matrix

$$X(t) = \begin{bmatrix} 2\,e^{2t} & e^{-t} \\ e^{2t} & 2\,e^{-t} \end{bmatrix}.$$

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Fundamental solutions to homogeneous systems.

Example

Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\,e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

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that is, $w(t) = 3\,e^{t}$.

Fundamental solutions to homogeneous systems.

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Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\,e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

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that is, $w(t) = 3\,e^{t}$. Hence Since $w(t) \neq 0$ for $t \in \mathbb{R}$.

Fundamental solutions to homogeneous systems.

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Show that $\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2\,e^{2t} \\ e^{2t} \end{bmatrix},\ \mathbf{x}^{(2)} = \begin{bmatrix} e^{-t} \\ 2\,e^{-t} \end{bmatrix} \right\}$ is a fundamental set for the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

Solution: Recall: $\mathbf{x}^{(1)\prime}(t) = A\,\mathbf{x}^{(1)}(t)$, and $\mathbf{x}^{(2)\prime}(t) = A\,\mathbf{x}^{(2)}(t)$.

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We conclude: The solutions form a fundamental set. $\lhd$

Properties of differential linear systems (Sect. 7.4)

- Review: $n \times n$ linear differential systems.
- Fundamental solutions to homogeneous systems.
- **Existence and uniqueness of solutions to IVP.**
- The Wronskian of n solutions.

Existence and uniqueness of solutions to IVP.

Theorem (Existence and uniqueness)

If the $n \times n$ matrix-valued function A and the n-vector $\mathbf{b}$ are continuous on $[t_0, t_1] \subset \mathbb{R}$, then the linear system

$$\mathbf{x}'(t) = A(t)\,\mathbf{x} + \mathbf{b}(t) \tag{2}$$

always has a fundamental set of solutions

$$\{\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)\}.$$

Furthermore, the initial value problem given by Eq. (2) together with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution.

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Remarks:

- The initial value problem contains n initial conditions.

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Furthermore, the initial value problem given by Eq. (2) together with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution.

Remarks:

- The initial value problem contains n initial conditions.
- We will study how to obtain such solutions in the case of constant coefficients systems, $A(t) = A_0$.

Existence and uniqueness of solutions to IVP.

Example

Find the solution to the IVP

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

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Solution: We need to find a fundamental set of solutions.
From the previous Example: A fundamental set is

$$\left\{ \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t},\ \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} \right\}.$$

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$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

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That is $\mathbf{x}(t) = X(t)\,\mathbf{c}$.

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That is $\mathbf{x}(t) = X(t)\,\mathbf{c}$. The initial condition: $\mathbf{x}(0) = X(0)\,\mathbf{c}$.

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Existence and uniqueness of solutions to IVP.

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$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

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$$\mathbf{c} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}(t) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}. \qquad \triangleleft$$

Existence and uniqueness of solutions to IVP.

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(a) Next class we learn how to obtain solutions to $\mathbf{x}' = A\,\mathbf{x}$ with A constant.

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The eigenvalue eigenvector pairs for A are:

$$\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

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(c) For example: $\mathbf{x}' = A\,\mathbf{x}$, with $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$.

The eigenvalue eigenvector pairs for A are:

$$\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_2 = -1, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Existence and uniqueness of solutions to IVP.

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(d) Next class we generalize the result of this example.

Properties of differential linear systems (Sect. 7.4)

- Review: $n \times n$ linear differential systems.
- Fundamental solutions to homogeneous systems.
- Existence and uniqueness of solutions to IVP.
- **The Wronskian of n solutions.**

The Wronskian of n solutions.

Theorem (Generalization of Abel result)

If A is an $n \times n$ continuous matrix-valued function, and $\mathbf{x}^{(i)}$, with $i = 1, \cdots, n$, are arbitrary solutions of the differential equation $\mathbf{x}' = A(t)\mathbf{x}$, then the Wronskian

$$w(t) = \det\big(X(t)\big), \qquad X(t) = \big[\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)\big]$$

satisfies the equation

$$w(t) = w(t_0)\, e^{\alpha(t)}, \qquad \alpha(t) = \int_{t_0}^{t} \operatorname{tr} A(\tau)\, d\tau.$$

where $\operatorname{tr}(A)(t) = a_{11}(t) + \cdots + a_{nn}(t)$.

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Remark: If the Wronskian $w(t_2) \neq 0$ at a single point $t_2 \in [t_0, t_1]$, then $w(t) \neq 0$ for all $t \in [t_0, t_1]$.

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Example

Verify the generalized Abel Theorem for a fundamental set of solutions to

$$\mathbf{x} = A\,\mathbf{x}, \quad A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}.$$

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that is, $w(t) = 3\,e^{t}$. Notice: $w(0) = 3$ and $\mathrm{tr}\,(A) = 3 - 2 = 1$. Therefore,

$$w(t) = w(0)\,e^{\mathrm{tr}\,(A)\,t}. \qquad \triangleleft$$

Real, distinct eigenvalues (Sect. 7.5)

- Review: $n \times n$ linear differential systems.
- The case of diagonalizable matrices.
- Examples: 2×2 linear systems.
- Classification of 2×2 diagonalizable systems.
- Phase portraits for 2×2 systems.

Review: $n \times n$ linear differential systems.

Recall:

- Given an $n \times n$ matrix $A(t)$, n-vector $\mathbf{b}(t)$, find $\mathbf{x}(t)$ solution
$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$

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$$\mathbf{x}'(t) = A\,\mathbf{x}(t) + \mathbf{b}(t).$$
- We study homogeneous, constant coefficient systems, that is,
$$\mathbf{x}'(t) = A\,\mathbf{x}(t).$$

Review: $n \times n$ linear differential systems.

Recall:

- Given continuous functions A, $\mathbf{b}$ on $(t_1, t_2) \subset \mathbb{R}$, a constant $t_0 \in (t_1, t_2)$ and a vector $\mathbf{x}_0$, there exists a unique function $\mathbf{x}$ solution of the IVP

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- Today we learn to find such solution in the case of homogeneous, constant coefficients, $n \times n$ linear systems,

$$\mathbf{x}'(t) = A\,\mathbf{x}(t).$$

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- Review: $n \times n$ linear differential systems.
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- Examples: 2×2 linear systems.
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The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

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- We transform the system into a system for a variable $\mathbf{y}$ such that the system for $\mathbf{y}$ is decoupled,

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- We solve for $\mathbf{y}(t)$ and we transform back to $\mathbf{x}(t)$.

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Proof: Recall: $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, and $\mathbf{y}(t) = \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$.

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$$\mathbf{x}(t) = P\,\mathbf{y}(t) = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big] \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$$

The case of diagonalizable matrices.

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Remark:

- $A\,\mathbf{v}_i = \lambda_i\mathbf{v}_i$.
- The eigenvalues and eigenvectors of A are crucial to solve the differential linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$.

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Remark: Here is another argument useful to understand why the vector $\mathbf{x}(t) = c_1 \mathbf{v}_1 \, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n \, e^{\lambda_n t}$ is solution of the linear system $\mathbf{x}'(t) = A \, \mathbf{x}(t)$.

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Remark: Unlike the proof of the Theorem, this second argument does not show that $\mathbf{x}(t) = c_1\mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n\, e^{\lambda_n t}$ are all possible solutions to the system.

Real, distinct eigenvalues (Sect. 7.5)

- Review: $n \times n$ linear differential systems.
- The case of diagonalizable matrices.
- **Examples: 2×2 linear systems.**
- Classification of 2×2 diagonalizable systems.
- Phase portraits for 2×2 systems.

Examples: 2×2 linear systems.

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Fundamental solutions are

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

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$\triangleleft$

Examples: 2×2 linear systems.

Remark:

Re-writing the solution vector $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ in components $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$,

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Examples: 2×2 linear systems.

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Solve the IVP $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Therefore, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

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Therefore, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, hence $\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$. $\triangleleft$

Real, distinct eigenvalues (Sect. 7.5)

- Review: $n \times n$ linear differential systems.
- The case of diagonalizable matrices.
- Examples: 2×2 linear systems.
- **Classification of 2×2 diagonalizable systems.**
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Classification of 2×2 diagonalizable systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

Classification of 2×2 diagonalizable systems.

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Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) Matrix A has two different, real eigenvalues $\lambda_1 \neq \lambda_2$, so it has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions). (Section 7.5)

Classification of 2×2 diagonalizable systems.

Remark:

Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) Matrix A has two different, real eigenvalues $\lambda_1 \neq \lambda_2$, so it has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions). (Section 7.5)

(b) Matrix A has two different, complex eigenvalues $\lambda_1 = \overline{\lambda}_2$, so it has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$. (Section 7.6)

Classification of 2×2 diagonalizable systems.

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Remark:

(c-2) We will also study in Section 7.8 how to obtain solutions to a 2×2 system $\mathbf{x}' = A\,\mathbf{x}$ in the case that A is not diagonalizable and A has only one eigen-direction.

Real, distinct eigenvalues (Sect. 7.5)

- Review: $n \times n$ linear differential systems.
- The case of diagonalizable matrices.
- Examples: 2×2 linear systems.
- Classification of 2×2 diagonalizable systems.
- **Phase portraits for 2×2 systems.**

Phase portraits for 2×2 systems.

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- Case (ii): Express the solution as a vector-valued function,
$$\mathbf{x}(t) = c_1\, \mathbf{v}_1\, e^{\lambda_1 t} + c_2\, \mathbf{v}_2\, e^{\lambda_2 t},$$
  and plot the vector $\mathbf{x}(t)$ for different values of t.

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- Case (ii) is called a *phase portrait*.

Phase portraits for 2×2 systems.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

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Solution:

We start plotting the vectors

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

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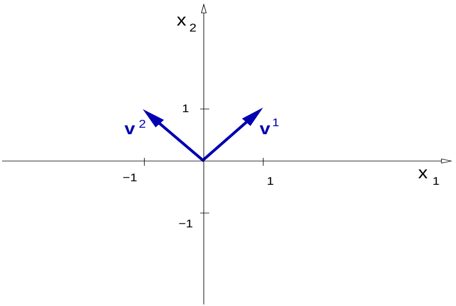

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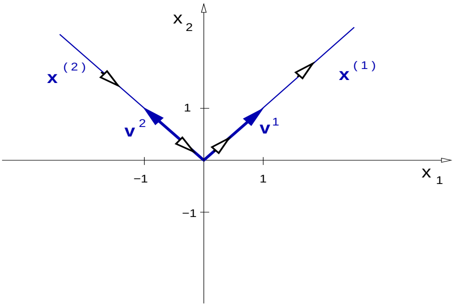

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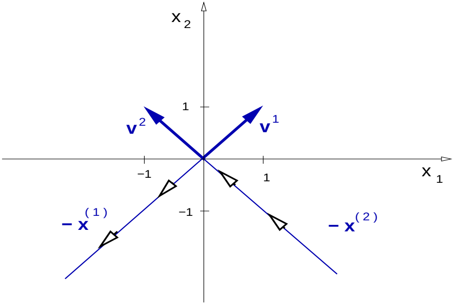

Phase portraits for 2 × 2 systems.

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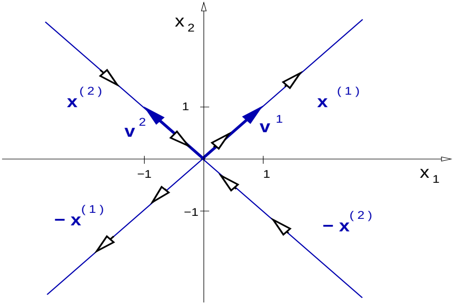

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$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)}, \quad \mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)},$$

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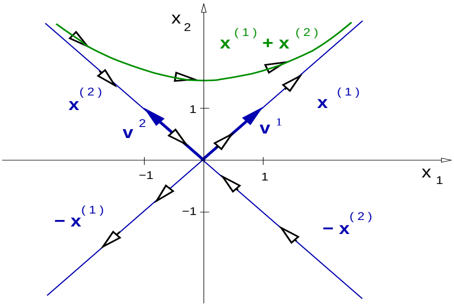

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Solution:

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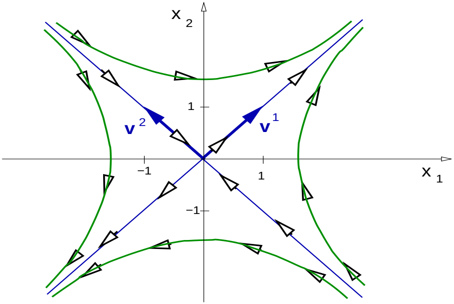

Phase portraits for 2×2 systems.

Problem:

Case (a): Consider a 2×2 matrix A having two different, real eigenvalues $\lambda_1 \neq \lambda_2$, so A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions).

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Given a solution $\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$, to $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, plot different solution vectors $\mathbf{x}(t)$ on the plane as function of t for different choices of the constants c_1 and c_2.

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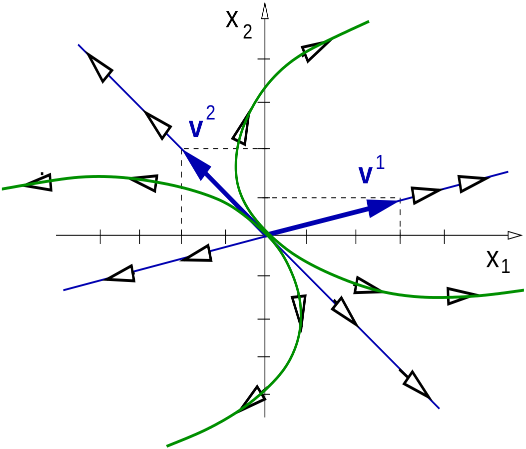
(iii) $\lambda_2 < \lambda_1 < 0$, both negative.

Phase portraits for 2×2 systems.

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $0 < \lambda_2 < \lambda_1$, both eigenvalue positive.



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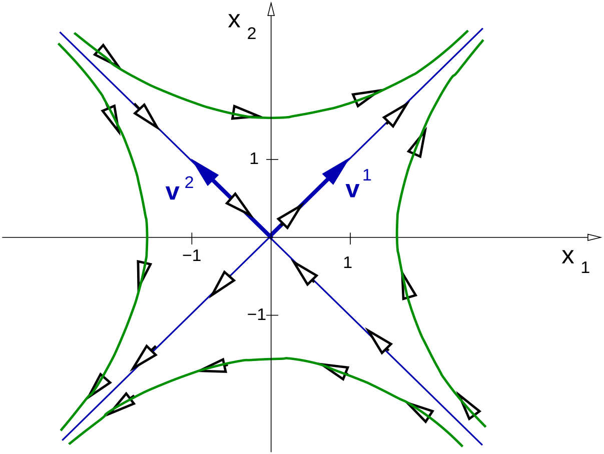

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