

## Review for Exam 2.

- ▶ 5 or 6 problems.
- ▶ No multiple choice questions.
- ▶ No notes, no books, no calculators.
- ▶ Problems similar to homeworks.
- ▶ Exam covers:
  - ▶ Regular-singular points (5.5).
  - ▶ Euler differential equation (5.4).
  - ▶ Power series solutions (5.2).
  - ▶ Variation of parameters (3.6).
  - ▶ Undetermined coefficients (3.5)
  - ▶ Constant coefficients, homogeneous, (3.1)-(3.4).

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## Regular-singular points (5.5).

Summary:

- ▶ Look for solutions  $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)}$ .

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► Recall: Since  $r \neq 0$ , holds

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► Find the indicial equation for  $r$ , the recurrence relation for  $a_n$ .

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(a) If  $(r_+ - r_-)$  is **not** an integer, then each  $r_+$  and  $r_-$  define linearly independent solutions.

(b) If  $(r_+ - r_-)$  is an integer, then both  $r_+$  and  $r_-$  define proportional solutions.



## Regular-singular points (5.5).

### Example

Consider the equation  $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$ . Use a power series centered at the regular-singular point  $x_0 = 0$  to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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$$x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)}$$

We also need to compute

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

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Re-label  $m = n + 2$  in the first term and then switch back to  $n$ ,

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Re-label  $m = n + 2$  in the first term and then switch back to  $n$ ,

$$\left(x^2 + \frac{1}{4}\right)y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$



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The equation is

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$

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$$\left[ r(r-1) + \frac{1}{4} \right] a_0 x^r + \left[ (r+1)r + \frac{1}{4} \right] a_1 x^{(r+1)} +$$

$$\sum_{n=2}^{\infty} \left[ (n+r)(n+r-1) a_n + a_{(n-2)} + \frac{1}{4} a_n \right] x^{(n+r)} = 0.$$

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The indicial equation  $r^2 - r + \frac{1}{4} = 0$  implies  $r_{\pm} = \frac{1}{2}$ .

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$$n^2 a_n = -a_{(n-2)} \Rightarrow a_n = -\frac{a_{(n-2)}}{n^2} \Rightarrow \begin{cases} a_2 = -\frac{a_0}{4}, \\ a_4 = -\frac{a_2}{16} \end{cases}$$

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**Solution:**  $r = \frac{1}{2}$ ,  $a_1 = 0$ ,  $a_2 = -\frac{a_0}{4}$ , and  $a_4 = \frac{a_0}{64}$ . Then,

$$y(x) = x^r (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots).$$

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$$y(x) = x^r (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots).$$

Recall:  $a_1 = 0$  and the recurrence relation imply  $a_n = 0$  for  $n$  odd.

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**Solution:**  $r = \frac{1}{2}$ ,  $a_1 = 0$ ,  $a_2 = -\frac{a_0}{4}$ , and  $a_4 = \frac{a_0}{64}$ . Then,

$$y(x) = x^r (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots).$$

Recall:  $a_1 = 0$  and the recurrence relation imply  $a_n = 0$  for  $n$  odd. Therefore,

$$y(x) = a_0 x^{1/2} \left(1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 + \cdots\right).$$



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# Euler differential equation (5.4).

Summary:

$$\blacktriangleright (x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0.$$

# Euler differential equation (5.4).

## Summary:

- ▶  $(x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0.$
- ▶ Find  $r_{\pm}$  solutions of  $r(r - 1) + p_0 r + q_0 = 0.$

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## Summary:

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## Review for Exam 2.

- ▶ 5 problems.
- ▶ No multiple choice questions.
- ▶ No notes, no books, no calculators.
- ▶ Problems similar to homeworks.
- ▶ Exam covers:
  - ▶ Regular-singular points (5.5).
  - ▶ Euler differential equation (5.4).
  - ▶ Power series solutions (5.2).
  - ▶ Variation of parameters (3.6).
  - ▶ **Undetermined coefficients (3.5)**
  - ▶ Constant coefficients, homogeneous, (3.1)-(3.4).

## Undetermined coefficients (3.5)

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We guess:  $y_{p_2} = k e^{3x}$ . Then,  $y''_{p_2} = 9 e^{3x}$ ,

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Therefore, the general solution is

$$y(x) = c_1 \sin(2x) + \left(c_2 - \frac{3}{4}x\right) \cos(2x) + \frac{1}{13} e^{3x}. \quad \triangleleft$$

# The Laplace Transform (Sect. 6.1).

- ▶ The definition of the Laplace Transform.
- ▶ Review: Improper integrals.
- ▶ Examples of Laplace Transforms.
- ▶ A table of Laplace Transforms.
- ▶ Properties of the Laplace Transform.
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# The definition of the Laplace Transform.

## Definition

The function  $F : D_F \rightarrow \mathbb{R}$  is the *Laplace transform* of a function  $f : [0, \infty) \rightarrow \mathbb{R}$  iff for all  $s \in D_F$  holds,

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

where  $D_F \subset \mathbb{R}$  is the set where the integral converges.



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The function  $F : D_F \rightarrow \mathbb{R}$  is the *Laplace transform* of a function  $f : [0, \infty) \rightarrow \mathbb{R}$  iff for all  $s \in D_F$  holds,

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

where  $D_F \subset \mathbb{R}$  is the set where the integral converges.

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- ▶ The Laplace transform is also a function:  $f \mapsto \mathcal{L}[f]$ .

# The Laplace Transform (Sect. 6.1).

- ▶ The definition of the Laplace Transform.
- ▶ **Review: Improper integrals.**
- ▶ Examples of Laplace Transforms.
- ▶ A table of Laplace Transforms.
- ▶ Properties of the Laplace Transform.
- ▶ Laplace Transform and differential equations.

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In other words,  $F(s) = \mathcal{L}[1]$  is the function  $F : D_F \rightarrow \mathbb{R}$  given by

$$f(t) = 1, \quad F(s) = \frac{1}{s}, \quad D_F = (0, \infty). \quad \triangleleft$$



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Integrating by parts twice it is not difficult to obtain:

$$\begin{aligned} & \int_0^N e^{-st} \sin(at) dt = \\ & -\frac{1}{s} [e^{-st} \sin(at)] \Big|_0^N - \frac{a}{s^2} [e^{-st} \cos(at)] \Big|_0^N - \frac{a^2}{s^2} \int_0^N e^{-st} \sin(at) dt. \end{aligned}$$

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This identity implies

$$\left(1 + \frac{a^2}{s^2}\right) \int_0^N e^{-st} \sin(at) dt = -\frac{1}{s} [e^{-st} \sin(at)] \Big|_0^N - \frac{a}{s^2} [e^{-st} \cos(at)] \Big|_0^N.$$

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Hence, it is not difficult to see that

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which is equivalent to

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \quad s > 0.$$

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- ▶ **A table of Laplace Transforms.**
- ▶ Properties of the Laplace Transform.
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# A table of Laplace Transforms.

$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0,$
$f(t) = e^{at}$	$F(s) = \frac{1}{s - a}$	$s > \max\{a, 0\},$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0,$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2 + a^2}$	$s > 0,$
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2 + a^2}$	$s > 0,$
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$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s - a)^{(n+1)}}$	$s > \max\{a, 0\},$
$f(t) = e^{at} \sin(bt)$	$F(s) = \frac{b}{(s - a)^2 + b^2}$	$s > \max\{a, 0\}.$

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# Properties of the Laplace Transform.

## Theorem (Sufficient conditions)

*If the function  $f : [0, \infty) \rightarrow \mathbb{R}$  is piecewise continuous and there exist positive constants  $k$  and  $a$  such that*

$$|f(t)| \leq k e^{at},$$

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*If the  $\mathcal{L}[f]$  and  $\mathcal{L}[g]$  are well-defined and  $a, b$  are constants, then*

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We then conclude that  $\mathcal{L}[f'] = s \mathcal{L}[f] - f(0)$ .

# The Laplace Transform (Sect. 6.1).

- ▶ The definition of the Laplace Transform.
- ▶ Review: Improper integrals.
- ▶ Examples of Laplace Transforms.
- ▶ A table of Laplace Transforms.
- ▶ Properties of the Laplace Transform.
- ▶ **Laplace Transform and differential equations.**

# Laplace Transform and differential equations.

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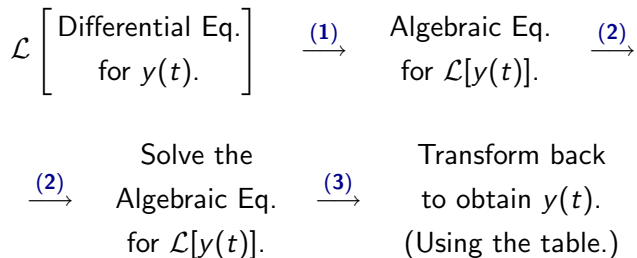
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# The Laplace Transform and the IVP (Sect. 6.2).

- ▶ Solving differential equations using  $\mathcal{L}[ \ ]$ .
  - ▶ Homogeneous IVP.
  - ▶ First, second, higher order equations.
  - ▶ Non-homogeneous IVP.

# Solving differential equations using $\mathcal{L}[\ ]$ .

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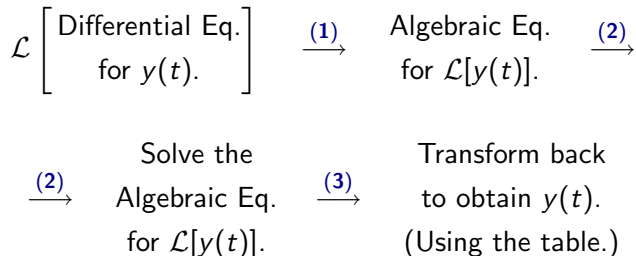
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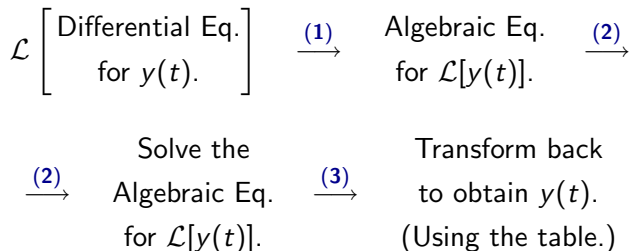
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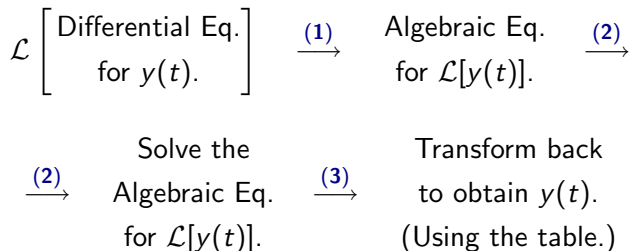
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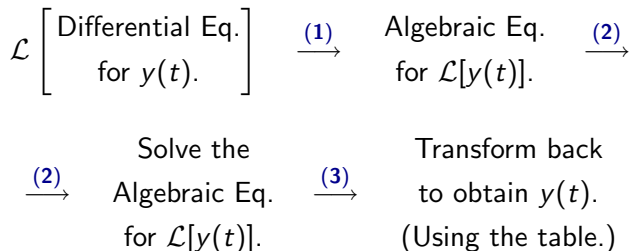


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# The Laplace Transform and the IVP (Sect. 6.2).

- ▶ Solving differential equations using  $\mathcal{L}[ \ ]$ .
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Use the Laplace transform to find the solution  $y(t)$  to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

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Derivatives are transformed into power functions,

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Therefore, we rewrite:  $\mathcal{L}[y] = \frac{(s-1)}{(s-2)(s+1)}.$

Find constants  $a$  and  $b$  such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

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Hence,  $a = \frac{1}{3}$  and  $b = \frac{2}{3}$ . Then,  $\mathcal{L}[y] = \frac{1}{3} \frac{1}{(s-2)} + \frac{2}{3} \frac{1}{(s+1)}.$

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We conclude that:  $y(t) = \frac{1}{3}(e^{2t} + 2e^{-t})$ .



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We obtain:  $\mathcal{L}[y] = \frac{(s - 3)}{(s - 2)^2}$ .

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### Example

Use the Laplace transform to find the solution  $y(t)$  to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

**Solution:** Recall:  $\mathcal{L}[y] = \frac{(s-3)}{(s-2)^2}$ . We find the partial fraction,

$$\frac{(s-3)}{(s-2)^2} = \frac{a}{(s-2)} + \frac{b}{(s-2)^2} \Rightarrow s-3 = a(s-2) + b$$

If  $s = 2$ ,

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We conclude that  $y(t) = e^{2t} - te^{2t}$ .



# The Laplace Transform and the IVP (Sect. 6.2).

- ▶ Solving differential equations using  $\mathcal{L}[ \ ]$ .
  - ▶ Homogeneous IVP.
  - ▶ **First, second, higher order equations.**
  - ▶ Non-homogeneous IVP.

## First, second, higher order equations.

### Example

Use the Laplace Transform to find the solution of  $y^{(4)} - 4y = 0$ ,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

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We obtain,  $\mathcal{L}[y] = \frac{s^3 - 2s}{(s^4 - 4)}.$

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We conclude that  $y(t) = \cos(\sqrt{2} t)$ .



# The Laplace Transform and the IVP (Sect. 6.2).

- ▶ Solving differential equations using  $\mathcal{L}[ \ ]$ .
  - ▶ Homogeneous IVP.
  - ▶ First, second, higher order equations.
  - ▶ **Non-homogeneous IVP.**

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## Example

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$$y'' - 4y' + 4y = 3\sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

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Introduce this source term in the differential equation,

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Introduce this source term in the differential equation,

$$\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

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Derivatives are transformed into power functions,

$$\left[ s^2 \mathcal{L}[y] - s y(0) - y'(0) \right] - 4 \left[ s \mathcal{L}[y] - y(0) \right] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$$



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Rewrite the above equation,

$$(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0) + \frac{6}{s^2 + 4}.$$

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Introduce the initial conditions,

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Solution: Recall:  $(s^2 - 4s + 4) \mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}$ .

Therefore,  $\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4s + 4)(s^2 + 4)}$ .

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**Solution:** Recall:  $(s^2 - 4s + 4)\mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}$ .

$$\text{Therefore, } \mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4s + 4)(s^2 + 4)}.$$

From an Example above:  $s^2 - 4s + 4 = (s - 2)^2$ ,

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We obtain the system

$$\begin{aligned} a + c &= 0, & -4a + b - 2c + d &= 0, \\ 4a - 4b + 4c &= 0, & 4b - 8c + 4d &= 6. \end{aligned}$$

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We conclude that

$$y(t) = (1-t)e^{2t} + \frac{3}{8}(2t-1)e^{2t} + \frac{3}{8}\cos(2t). \quad \triangleleft$$