Review for Exam 2.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - Euler differential equation (5.4).
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - Undetermined coefficients (3.5)
  - Constant coefficients, homogeneous, (3.1)-(3.4).

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Regular-singular points (5.5).

Summary:

- Look for solutions $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)}$.

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- Recall: Since $r \neq 0$, holds

$$y' = \sum_{n=0}^{\infty}(n+r)a_n(x-x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty}(n+r)a_n(x-x_0)^{(n+r-1)},$$

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- Introduce the larger root r_+ of the indicial polynomial into the recurrence relation and solve for a_n.

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- Find the indicial equation for r, the recurrence relation for a_n.
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  (a) If $(r_+ - r_-)$ is not an integer, then each r_+ and r_- define linearly independent solutions.

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- Find the indicial equation for r, the recurrence relation for a_n.
- Introduce the larger root r_+ of the indicial polynomial into the recurrence relation and solve for a_n.
  (a) If $(r_+ - r_-)$ is not an integer, then each r_+ and r_- define linearly independent solutions.
  (b) If $(r_+ - r_-)$ is an integer, then both r_+ and r_- define proportional solutions.

Regular-singular points (5.5).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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Solution: $y = \sum_{n=0}^{\infty} a_n x^{(n+r)},$

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Solution: $y = \sum_{n=0}^{\infty} a_n x^{(n+r)}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r-2)}$,

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We also need to compute

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

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Re-label $m = n + 2$ in the first term and then switch back to n,

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Re-label $m = n + 2$ in the first term and then switch back to n,

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

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The equation is

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$

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$$\left[r(r-1) + \frac{1}{4}\right] a_0\, x^r + \left[(r+1)r + \frac{1}{4}\right] a_1\, x^{(r+1)} +$$

$$\sum_{n=2}^{\infty}\left[(n+r)(n+r-1)a_n + a_{(n-2)} + \frac{1}{4}a_n\right] x^{(n+r)} = 0.$$

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The indicial equation $r^2 - r + \dfrac{1}{4} = 0$ implies $r_\pm = \dfrac{1}{2}$.

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The indicial equation $r^2 - r + \dfrac{1}{4} = 0$ implies $r_\pm = \dfrac{1}{2}$.

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$$\left[\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)+\frac{1}{4}\right] a_n = -a_{(n-2)}$$

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$$n^2 a_n = -a_{(n-2)} \;\Rightarrow\; a_n = -\frac{a_{(n-2)}}{n^2}$$

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Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

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$$n^2 a_n = -a_{(n-2)} \;\Rightarrow\; a_n = -\frac{a_{(n-2)}}{n^2} \;\Rightarrow\; \begin{cases} a_2 = -\dfrac{a_0}{4}, \\ a_4 = -\dfrac{a_2}{16} \end{cases}$$

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Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \dfrac{1}{2}$, $a_1 = 0$, $a_2 = -\dfrac{a_0}{4}$, and $a_4 = \dfrac{a_0}{64}$.

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Solution: $r = \dfrac{1}{2}$, $a_1 = 0$, $a_2 = -\dfrac{a_0}{4}$, and $a_4 = \dfrac{a_0}{64}$. Then,

$$y(x) = x^r\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right).$$

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Recall: $a_1 = 0$ and the recurrence relation imply $a_n = 0$ for n odd.

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Recall: $a_1 = 0$ and the recurrence relation imply $a_n = 0$ for n odd. Therefore,

$$y(x) = a_0x^{1/2}\big(1 - \frac{1}{4}\,x^2 + \frac{1}{64}\,x^4 + \cdots\big).$$

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  - Constant coefficients, homogeneous, (3.1)-(3.4).

Euler differential equation (5.4).

Summary:

- $(x - x_0)^2\, y'' + (x - x_0) p_0\, y' + q_0\, y = 0.$

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- $(x - x_0)^2\, y'' + (x - x_0) p_0\, y' + q_0\, y = 0.$
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- Find $r_\pm$ solutions of $r(r-1) + p_0 r + q_0 = 0.$
- If $r_+ \neq r_-$ and both are real, then fundamental solutions are

$$y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_-}.$$

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- If $r_\pm = \alpha \pm i\beta$, then real-valued fundamental solutions are

$$y_+ = |x - x_0|^{\alpha} \cos\big(\beta \ln |x - x_0|\big),\; y_- = |x - x_0|^{\alpha} \sin\big(\beta \ln |x - x_0|\big).$$

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$$y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_+} \ln |x - x_0|.$$

Euler differential equation (5.4).

Example

Find real-valued fundamental solutions of

$$(x-2)^2\, y'' + 5(x-2)\, y' + 8\, y = 0.$$

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$$y_+(x) = |x-2|^{-2} \cos\bigl(2 \ln |x-2|\bigr),$$

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$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

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Variation of parameters (3.6).

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Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

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Review for Exam 2.

- 5 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - Euler differential equation (5.4).
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - **Undetermined coefficients (3.5)**
  - Constant coefficients, homogeneous, (3.1)-(3.4).

Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}$$

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Solution: Recall: $y_1 = \sin(2x)$, and $y_2 = \cos(2x)$.

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$$(9+4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13} \quad \Rightarrow \quad y_{p_2} = \frac{1}{13}\,e^{3x}.$$

Therefore, the general solution is

$$y(x) = c_1\sin(2x) + \left(c_2 - \frac{3}{4}\,x\right)\cos(2x) + \frac{1}{13}\,e^{3x}. \qquad \triangleleft$$

The Laplace Transform (Sect. 6.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
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The definition of the Laplace Transform.

Definition

The function $F : D_F \to \mathbb{R}$ is the *Laplace transform* of a function $f : [0, \infty) \to \mathbb{R}$ iff for all $s \in D_F$ holds,

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- The integral converges iff the limit exists.

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Since $\displaystyle\lim_{N\to\infty} e^{-aN} = 0$ for $a > 0$, we conclude $\displaystyle\int_0^{\infty} e^{-at}\,dt = \frac{1}{a}.$ $\lhd$

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In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \to \mathbb{R}$ given by

$$f(t) = 1, \qquad F(s) = \frac{1}{s}, \qquad D_F = (0, \infty). \qquad \triangleleft$$

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$$f(t) = e^{at}, \qquad F(s) = \frac{1}{(s-a)}, \qquad s > a. \qquad \triangleleft$$

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This identity implies

$$\left(1+\frac{a^2}{s^2}\right)\int_0^N e^{-st}\sin(at)\, dt = -\frac{1}{s}\left[e^{-st}\sin(at)\right]\Big|_0^N - \frac{a}{s^2}\left[e^{-st}\cos(at)\right]\Big|_0^N.$$

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Hence, it is not difficult to see that

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A table of Laplace Transforms.

$$f(t) = 1 \qquad F(s) = \frac{1}{s} \qquad s > 0,$$

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Theorem (Sufficient conditions)

If the function $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and there exist positive constants k and a such that

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where we used that $\lim_{n\to\infty} e^{-sn} f(n) = 0$ for s big enough, and we also used that $\mathcal{L}[f]$ is well-defined.

We then conclude that $\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0)$.

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Laplace Transform and differential equations.

Remark: Laplace Transforms can be used to find solutions to differential equations with constant coefficients.

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Laplace Transform and differential equations.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \qquad y(0) = 3.$$

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Find an algebraic equation for $\mathcal{L}[y]$.

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Also recall the property: $\mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0)$, that is,

$$\Big[s\,\mathcal{L}[y] - y(0)\Big] + 2\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s+2)\mathcal{L}[y] = y(0).$$

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$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \quad \Rightarrow \quad y(t) = 3e^{-2t}$. $\triangleleft$

The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - Non-homogeneous IVP.

Solving differential equations using $\mathcal{L}[\ ]$.

Remark: The method works with:

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Recall:

(a) $\mathcal{L}[a\,f(t) + b\,g(t)] = a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)]$;

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(b) $\mathcal{L}\left[y^{(n)}\right] = s^n\,\mathcal{L}[y] - s^{(n-1)}\,y(0) - s^{(n-2)}\,y'(0) - \cdots - y^{(n-1)}(0)$.

The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - **Homogeneous IVP.**
  - First, second, higher order equations.
  - Non-homogeneous IVP.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

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Solution: Compute the $\mathcal{L}[\;]$ of the differential equation,

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The $\mathcal{L}[\ ]$ is a linear function,

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Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - \Big[s\,\mathcal{L}[y] - y(0)\Big] - 2\,\mathcal{L}[y] = 0,$$

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We the obtain $(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0)$.

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Differential equation for y $\xrightarrow{\mathcal{L}[\ ]}$ Algebraic equation for $\mathcal{L}[y]$.

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Introduce the initial condition,

Homogeneous IVP.

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$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1).$$

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We can solve for the unknown $\mathcal{L}[y]$ as follows,

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The partial fraction method: Find the zeros of the denominator,

$$s^2 - s - 2 = 0$$

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Therefore, we rewrite: $\mathcal{L}[y] = \dfrac{(s-1)}{(s-2)(s+1)}$.

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Solution: Recall: $\mathcal{L}[y] = \dfrac{(s-1)}{(s^2 - s - 2)}$.

The partial fraction method: Find the zeros of the denominator,

$$s^2 - s - 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\left[1 \pm \sqrt{1+8}\right] \quad \Rightarrow \quad \begin{cases} s_+ = 2, \\ s_- = -1, \end{cases}$$

Therefore, we rewrite: $\mathcal{L}[y] = \dfrac{(s-1)}{(s-2)(s+1)}$.

Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Recall: $\dfrac{(s-1)}{(s-2)(s+1)} = \dfrac{a}{s-2} + \dfrac{b}{s+1}.$

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A simple calculation shows

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$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$$

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$$(s-1) = s(a+b) + (a-2b)$$

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$$(s-1) = s(a+b) + (a-2b) \quad \Rightarrow \quad \begin{cases} a + b = 1, \\ a - 2b = -1 \end{cases}$$

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Hence, $a = \dfrac{1}{3}$ and $b = \dfrac{2}{3}$.

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$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

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$$(s-1) = s(a+b) + (a-2b) \quad \Rightarrow \quad \begin{cases} a + b = 1, \\ a - 2b = -1 \end{cases}$$

Hence, $a = \dfrac{1}{3}$ and $b = \dfrac{2}{3}$. Then, $\mathcal{L}[y] = \dfrac{1}{3}\,\dfrac{1}{(s-2)} + \dfrac{2}{3}\,\dfrac{1}{(s+1)}$.

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$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

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Solution: Recall: $\mathcal{L}[y] = \frac{1}{3}\,\frac{1}{(s-2)} + \frac{2}{3}\,\frac{1}{(s+1)}$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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$$\mathcal{L}[y] = \frac{1}{3}\,\mathcal{L}[e^{2t}] + \frac{2}{3}\,\mathcal{L}[e^{-t}] = \mathcal{L}\Big[\frac{1}{3}\big(e^{2t} + 2e^{-t}\big)\Big]$$

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We conclude that: $y(t) = \frac{1}{3}\big(e^{2t} + 2e^{-t}\big)$. $\triangleleft$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Homogeneous IVP.

Example

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$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

Homogeneous IVP.

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$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$

The $\mathcal{L}[\ ]$ is a linear function,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = 0.$$

Homogeneous IVP.

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Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = 0,$$

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Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = 0,$$

Therefore, $(s^2 - 4s + 4)\,\mathcal{L}[y] = (s - 4)\,y(0) + y'(0).$

Homogeneous IVP.

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Introduce the initial conditions,

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

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Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = (s - 4)\,y(0) + y'(0)$.

Introduce the initial conditions, $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3$.

Homogeneous IVP.

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Solve for $\mathcal{L}[y]$ as follows:

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Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)}$.

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Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)}$.

The partial fraction method: Find the roots of the denominator,

$s^2 - 4s + 4 = 0$

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$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\left[4 \pm \sqrt{16 - 16}\right]$$

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We obtain: $\mathcal{L}[y] = \dfrac{(s-3)}{(s-2)^2}$.

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If $s = 2$,

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If $s = 2$, then $b = -1$.

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If $s = 2$, then $b = -1$. If $s = 3$, then $a = 1$. Hence

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

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$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{(n+1)}}$$

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$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{(n+1)}} \quad \Rightarrow \quad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{1}{s-2} - \dfrac{1}{(s-2)^2}$ and

$$\frac{1}{s-2} = \mathcal{L}[e^{2t}], \qquad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Solution: Recall: $\mathcal{L}[y] = \dfrac{1}{s-2} - \dfrac{1}{(s-2)^2}$ and

$$\frac{1}{s-2} = \mathcal{L}[e^{2t}], \qquad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}]$$

Homogeneous IVP.

Example

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We conclude that $y(t) = e^{2t} - te^{2t}$. $\triangleleft$

The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - **First, second, higher order equations.**
  - Non-homogeneous IVP.

First, second, higher order equations.

Example

Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

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$$\big[s^4\,\mathcal{L}[y] - s^3 + 2s\big] - 4\,\mathcal{L}[y] = 0$$

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We obtain, $\mathcal{L}[y] = \dfrac{s^3 - 2s}{(s^4 - 4)}$.

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Solution: Recall: $\mathcal{L}[y] = \dfrac{s^3 - 2s}{(s^4 - 4)}$.

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)}$$

First, second, higher order equations.

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The last expression is in the table of Laplace Transforms,

$$\mathcal{L}[y] = \frac{s}{\left(s^2 + \left[\sqrt{2}\right]^2\right)}$$

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The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - **Non-homogeneous IVP.**

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Non-homogeneous IVP.

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Introduce this source term in the differential equation,

Non-homogeneous IVP.

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Introduce this source term in the differential equation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

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Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{6}{s^2+4}.$$

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Rewrite the above equation,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0) + \frac{6}{s^2+4}.$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \dfrac{6}{s^2 + 4}$.

Therefore, $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)} + \dfrac{6}{(s^2 - 4 + 4)(s^2 + 4)}$.

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From an Example above: $s^2 - 4s + 4 = (s - 2)^2$,

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From an Example above: $s^2 - 4s + 4 = (s-2)^2$,

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2} + \frac{6}{(s-2)^2(s^2+4)}.$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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From an Example above we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Use Partial fractions to simplify the last term above.

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Use Partial fractions to simplify the last term above.

Find constants a, b, c, d, such that

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{(s-2)} + \frac{d}{(s-2)^2}$$

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$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{(s-2)} + \frac{d}{(s-2)^2}$$

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{(as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)}{(s^2+4)(s-2)^2}$$

$$6 = (as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4).$$

Non-homogeneous IVP.

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$$6 = (as + b)(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4)$$

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$$6 = (as + b)(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4)$$

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$$6 = (as+b)(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2+4)$$

$$6 = a(s^3 - 4s^2 + 4s) + b(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4).$$

$$6 = (a+c)s^3 + (-4a + b - 2c + d)s^2 + (4a - 4b + 4c)s + (4b - 8c + 4d).$$

Non-homogeneous IVP.

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$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: $6 = (as + b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)$.

$$6 = (as+b)(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4)$$

$$6 = a(s^3 - 4s^2 + 4s) + b(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4).$$

$$6 = (a + c)s^3 + (-4a + b - 2c + d)s^2 + (4a - 4b + 4c)s + (4b - 8c + 4d).$$

We obtain the system

$$a + c = 0, \qquad -4a + b - 2c + d = 0,$$

$$4a - 4b + 4c = 0, \qquad 4b - 8c + 4d = 6.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: The solution for this linear system is

$$a = \frac{3}{8}, \qquad b = 0, \qquad c = -\frac{3}{8}, \qquad d = \frac{3}{4}.$$

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$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{3}{8}\,\frac{s}{s^2+4} - \frac{3}{8}\,\frac{1}{(s-2)} + \frac{3}{4}\,\frac{1}{(s-2)^2}.$$

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Use the table of Laplace Transforms

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{3}{8}\,\mathcal{L}[\cos(2t)] - \frac{3}{8}\,\mathcal{L}[e^{2t}] + \frac{3}{4}\,\mathcal{L}[te^{2t}].$$

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Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Summary: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \dfrac{6}{(s-2)^2(s^2+4)}$,

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Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

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We conclude that

$$y(t) = (1-t)\,e^{2t} + \frac{3}{8}\,(2t-1)\,e^{2t} + \frac{3}{8}\,\cos(2t).$$

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