Power series solutions near regular points (Sect. 5.2).

- We study: $P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$.
- Review of power series.
- Regular point equations.
- Solutions using power series.
- Examples of the power series method.

Review of power series.

Definition

The *power series* of a function $y : \mathbb{R} \to \mathbb{R}$ centered at $x_0 \in \mathbb{R}$ is

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^n.$$

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Remark: The Taylor series of $y(x) = \cos(x)$ centered at $x_0 = 0$ is

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{(2n)}.$$

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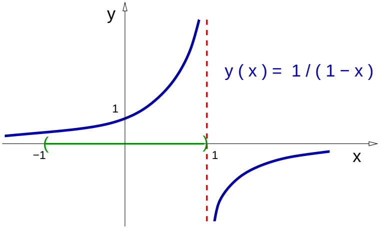

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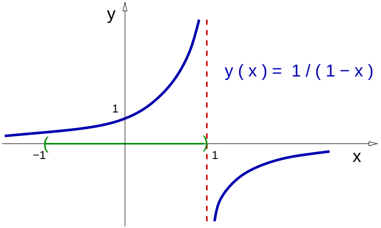

The power series

$$y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

converges only for $|x| < 1$.

◁

Review of power series.

Definition

The power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *converges absolutely* iff the series $\sum_{n=0}^{\infty} |a_n| \, |x - x_0|^n$ converges.

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Review of power series.

Definition

The *radius of convergence* of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is the number $\rho \geqslant 0$ that satisfies both

(a) the series converges absolutely for $|x - x_0| < \rho$;

(b) the series diverges for $|x - x_0| > \rho$.

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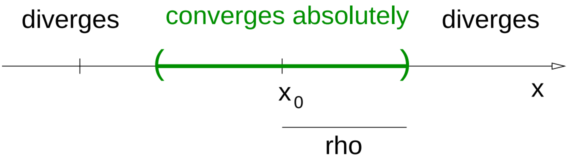

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Theorem (Ratio test)

Given the power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$*, introduce the number* $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$*. Then, the following statements hold:*

(1) *The power series converges in the domain* $|x - x_0| L < 1$*.*

(2) *The power series diverges in the domain* $|x - x_0| L > 1$*.*

(3) *The power series may or may not converge at* $|x - x_0| L = 1$*.*

Therefore, if $L \neq 0$*, then* $\rho = \frac{1}{L}$ *is the series radius of convergence; if* $L = 0$*, then the radius of convergence is* $\rho = \infty$*.*

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where $m = n - 1$, that is, $n = m + 1$.

Power series solutions near regular points (Sect. 5.2).

- We study: $P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$.
- Review of power series.
- **Regular point equations.**
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Regular point equations.

Problem: We look for solutions y of the variable coefficients equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0.$$

around $x_0 \in \mathbb{R}$ where $P(x_0) \neq 0$ using a power series representation of the solution centered at x_0, that is,

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Given continuous functions P, Q, $R : (x_1, x_2) \to \mathbb{R}$, a point $x_0 \in (x_1, x_2)$ is called a *regular point* of the equation

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Remark: The equation order does not change near regular points.

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(5) If possible, add up the resulting power series for the solution y.

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Examples of the power series method.

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$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

Solution: Recall: The solution is $y(x) = a_0\, e^{-c\,x}$.

We now use the power series method. We propose a power series centered at $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^n \quad \Rightarrow \quad y'(x) = \sum_{n=0}^{\infty} n a_n\, x^{(n-1)} = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)}.$$

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then, we conclude that the solution is $y(x) = a_0\,e^{-c\,x}$. $\triangleleft$

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We obtain: $a_{2k} = \dfrac{(-1)^k}{(2k)!}\, a_0$, for $k \geqslant 0$.

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$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}.$$

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The recurrence relation for the coefficients a_n is:

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$$y \simeq a_0 + a_1(x-2) + a_0(x-2)^2 + \left(\frac{a_0}{6} + \frac{a_1}{3}\right)(x-2)^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x-2)^4.$$

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$$y = a_0\Big[1 + (x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \cdots\Big]$$
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So the first three terms on each fundamental solution are given by

$$y_1 \simeq 1 + (x-2)^2 + \frac{1}{6}(x-2)^3, \quad y_2 \simeq (x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4.$$

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- We study the Euler Equation:
  $(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0.$
- Solutions to the Euler equation near x_0.
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.

Overview: Equations with singular points.

Recall: The point $x_0 \in \mathbb{R}$ is a singular point of the equation

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- The order of the differential equation changes in a neighborhood of a singular point.
- In the limit $x \to x_0$ the following could happen:
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  (b) Only one solution remains bounded.
  (c) None solution remains bounded.

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- Singular points where the singular behavior of the solution is somehow mild, in a sense to be made precise later, will be called regular-singular points.
- The main example of a equation with a regular-singular point is the Euler differential equation.

The Euler equation (Sect. 5.4).

- Overview: Equations with singular points.
- **We study the Euler Equation:**
  $(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0.$
- Solutions to the Euler equation near x_0.
- The roots of the indicial polynomial.
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Given real constants p_0, q_0, the *Euler differential equation* for the unknown y with singular point at $x_0 \in R$ is given by

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- The particular case $x_0 = 0$ is is given by

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This equation is called the indicial equation, and is also called the Euler characteristic equation.

Solutions to the Euler equation near x_0.

Theorem (Euler equation)

Given p_0, q_0, $x_0 \in \mathbb{R}$, consider the Euler equation

$$(x - x_0)^2 \, y'' + p_0 \, (x - x_0) \, y' + q_0 \, y = 0. \tag{3}$$

Let r_+, r_- be solutions of $r(r-1) + p_0 r + q_0 = 0$.

(a) *If $r_+ \neq r_-$, then a real-valued general solution of Eq. (3) is*

$$y(x) = c_0 |x - x_0|^{r_+} + c_1 |x - x_0|^{r_-}, \quad x \neq x_0, \quad c_0, \ c_1 \in \mathbb{R}.$$

(b) *If $r_+ = r_-$, then a real-valued general solution of Eq. (3) is*

$$y(x) = \Big[c_0 + c_1 \ln\big(|x - x_0|\big) \Big] |x - x_0|^{r_+}, \quad x \neq x_0, \quad c_0, \ c_1 \in \mathbb{R}.$$

Given $x_0 \neq x_1$, y_0, $y_1 \in \mathbb{R}$, there is a unique solution to the IVP

$$(x - x_0)^2 \, y'' + p_0 \, (x - x_0) \, y' + q_0 \, y = 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1.$$

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$$x^2\, y'' + 4x\, y' + 2\, y = 0.$$

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$$\big[r(r-1) - 3r + 13\big]\,x^r = 0$$

Different complex roots.

Example

Find the general solution of the Euler equation

$$x^2\,y'' - 3x\,y' + 13\,y = 0.$$

Solution: We look for solutions of the form $y(x) = x^r$,

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The solutions of the indicial equation $r^2 - 4r + 13 = 0$ are

$$r_{\pm} = \frac{1}{2}\big[4 \pm \sqrt{16 - 52}\big]$$

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The general solution is $y(x) = c_1\, |x|^{(2+3i)} + c_2\, |x|^{(2-3i)}$. $\lhd$

Different complex roots.

Theorem (Real-valued fundamental solutions)

If p_0, $q_0 \in \mathbb{R}$ satisfy that $\left[(p_0 - 1)^2 - 4q_0\right] < 0$, then the indicial polynomial $p(r) = r(r-1) + p_0 r + q_0$ of the Euler equation

$$x^2\, y'' + p_0 x\, y' + q_0\, y = 0 \tag{4}$$

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{(p_0 - 1)}{2}, \qquad \beta = \frac{1}{2}\sqrt{4q_0 - (p_0 - 1)^2}.$$

Furthermore, a fundamental set of solution to Eq. (4) is

$$\tilde{y}_1(x) = |x|^{(\alpha + i\beta)}, \qquad \tilde{y}_2(x) = |x|^{(\alpha - i\beta)},$$

while another fundamental set of solutions to Eq. (4) is

$$y_1(x) = |x|^{\alpha} \cos\big(\beta \ln|x|\big), \qquad y_2(x) = |x|^{\alpha} \sin\big(\beta \ln|x|\big).$$

Different complex roots.

Proof: Given $\tilde{y}_1 = |x|^{(\alpha+i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha-i\beta)}$,

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Proof: Given $\tilde{y}_1 = |x|^{(\alpha + i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha - i\beta)}$, introduce

$$y_1 = \frac{1}{2}\big(\tilde{y}_1 + \tilde{y}_2\big), \qquad y_1 = \frac{1}{2i}\big(\tilde{y}_1 - \tilde{y}_2\big).$$

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Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

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$$\tilde{y}_1 = |x|^{\alpha}\Big[\cos\big(\beta \ln|x|\big) + 1\sin\big(\beta \ln|x|\big)\Big],$$

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We conclude that

$$y_1(x) = |x|^{\alpha}\cos\big(\beta\,\ln|x|\big), \qquad y_2(x) = |x|^{\alpha}\sin\big(\beta\,\ln|x|\big).$$

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$$y(x) = c_1\,|x|^2 \cos\big(3\ln|x|\big) + c_2\,|x|^2 \sin\big(3\ln|x|\big), \quad c_1,\ c_2 \in \mathbb{R}. \quad \triangleleft$$

Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
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Equations with regular-singular points (Sect. 5.5).

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Recall:

The point $x_0 \in \mathbb{R}$ is a singular point of the equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

iff holds that $P(x_0) = 0$.

Equations with regular-singular points.

Definition

A singular point $x_0 \in \mathbb{R}$ of the equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

is called a *regular-singular* point iff the following limits are finite,

$$\lim_{x\to x_0} \frac{(x-x_0)\,Q(x)}{P(x)}, \qquad \lim_{x\to x_0} \frac{(x-x_0)^2\,R(x)}{P(x)},$$

and both functions

$$\frac{(x-x_0)\,Q(x)}{P(x)}, \qquad \frac{(x-x_0)^2\,R(x)}{P(x)},$$

admit convergent Taylor series expansions around x_0.

Equations with regular-singular points.

Remark:

- If x_0 is a regular-singular point of

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

and $P(x) \simeq (x - x_0)^n$ near x_0, then near x_0 holds

$$Q(x) \simeq (x - x_0)^{n-1}, \quad R(x) \simeq (x - x_0)^{n-2}.$$

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- The main example is an Euler equation, case $n = 2$,

$$(x - x_0)^2\,y'' + p_0(x - x_0)\,y' + q_0\,y = 0.$$

Equations with regular-singular points.

Example

Show that the singular point of every Euler equation is a regular-singular point.

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Solution: Consider the general Euler equation

$$(x - x_0)^2\, y'' + p_0(x - x_0)\, y' + q_0\, y = 0,$$

where p_0, q_0, x_0, are real constants.

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$$\lim_{x \to x_0} \frac{(x - x_0)\, Q(x)}{P(x)}$$

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Solution: Consider the general Euler equation

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We conclude that x_0 is a regular-singular point. $\triangleleft$

Equations with regular-singular points.

Remark: Every equation $Py'' + Qy' + Ry = 0$ with a regular-singular point at x_0 is close to an Euler equation.

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For $x \neq x_0$ divide the equation by $P(x)$,

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The factors between [ ] approach constants, say p_0, q_0, as $x \to x_0$,

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Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- **Examples: Equations with regular-singular points.**
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Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(1 - x^2)\, y'' - 2x\, y' + \alpha(\alpha + 1)\, y = 0,$$

where α is a real constant.

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Case $x_0 = 1$:

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We conclude that $x_0 = 1$ is a regular-singular point.

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Therefore, the point $x_1 = -1$ is a regular-singular point. $\lhd$

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$$\frac{(x-1)^2\,R(x)}{P(x)} = \frac{2(x-1)^2}{(x+2)^2(x-1)} = \frac{2(x-1)}{(x+2)^2};$$

Both functions have Taylor series around $x_1 = 1$.

Examples: Equations with regular-singular points.

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Find the regular-singular points of the differential equation

$$(x+2)^2(x-1)\,y'' + 3(x-1)\,y' + 2\,y = 0.$$

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Therefore, the point $x_1 = -1$ is a regular-singular point. $\triangleleft$

Examples: Equations with regular-singular points.

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$$x\,y'' - x\ln(|x|)\,y' + 3x\,y = 0.$$

Solution: Recall: $\displaystyle\lim_{x\to 0} \frac{xQ(x)}{P(x)} = 0$ and $\displaystyle\lim_{x\to 0} \frac{x^2R(x)}{P(x)} = 0$.

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and the log function does not have a Taylor series at $x_0 = 0$.

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and the log function does not have a Taylor series at $x_0 = 0$.

We conclude that $x_0 = 0$ is not a regular-singular point. $\lhd$

Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- **Method to find solutions.**
- Example: Method to find solutions.

Method to find solutions.

Recall: If x_0 is a regular-singular point of

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0,$$

with limits $\displaystyle\lim_{x \to x_0} \frac{(x - x_0) Q(x)}{P(x)} = p_0$ and $\displaystyle\lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)} = q_0$,

then the coefficients of the differential equation above near x_0 are close to the coefficients of the Euler equation

$$(x - x_0)^2\, y'' + p_0 (x - x_0)\, y' + q_0\, y = 0.$$

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Idea: If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.

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Idea: If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.

Recall: One solution of an Euler equation is $y(x) = (x-x_0)^r$.

Method to find solutions.

Summary: Solutions for equations with regular-singular points:

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(1) Look for a solution y of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^{(n+r)};$$

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(1) Look for a solution y of the form

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(1) Look for a solution y of the form

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(2) Introduce this power series expansion into the differential equation and find both a the exponent r and a recurrence relation for the coefficients a_n;

(3) First find the solutions for the constant r. Then, introduce this result for r into the recurrence relation for the coefficients a_n. Only then, solve this latter recurrence relation for the coefficients a_n.

Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
- **Example: Method to find solutions.**

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Example: Method to find solutions.

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Solution: We look for a solution $y(x) = \sum_{n=0}^{\infty} a_n\, x^{(n+r)}$.

The first and second derivatives are given by

$$y' = \sum_{n=0}^{\infty} (n+r) a_n\, x^{(n+r-1)},$$

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In the case $r = 0$ we had the relation

$$\sum_{n=0}^{\infty} n a_n\, x^{(n-1)} = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)},$$

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but for $r \neq 0$ this relation is not true.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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Solution: We now compute the term $(x+3)y$,

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$$(x+3)\, y = \sum_{n=1}^{\infty} a_{(n-1)}\, x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n\, x^{(n+r)}.$$

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$$-x(x+3)\,y' = -(x^2+3x)\sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r-1)}$$

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$$x^2\, y'' = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r-2)}$$

$$x^2\, y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r)}.$$

Example: Method to find solutions.

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The guiding principle to rewrite each term is to have the power function $x^{(n+r)}$ labeled in the same way on every term.

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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We split the sums into the term $n = 0$ and a sum containing the terms with $n \geqslant 1$,

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We split the sums into the term $n = 0$ and a sum containing the terms with $n \geqslant 1$, that is,

$$0 = \big[r(r-1) - 3r + 3\big]\, a_0 x^r +$$

$$\sum_{n=1}^{\infty}\Big[(n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n\Big]\, x^{(n+r)}$$

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Find the solution y near the regular-singular point $x_0 = 0$ of

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Solution: Therefore, $\big[r(r-1) - 3r + 3\big] = 0$ and

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The last expression can be rewritten as follows,

$$\Big[\big[(n+r)(n+r-1) - 3(n+r) + 3\big]\,a_n - (n+r-1-1)a_{(n-1)}\Big] = 0,$$

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Solution: Hence, the recurrence relation is given by the equations

$$r(r-1) - 3r + 3 = 0,$$
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Second: Introduce the first solution r_+ into the second equation above and solve for the a_n; the result is a solution y_+ of the original differential equation;

Third: Introduce the second solution r_- into into the second equation above and solve for the a_n;

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$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}\big[4 \pm \sqrt{16-12}\big]$$

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One can check that the solution y_+ is

$$y_+ = a_0\, x^3 \Big[1 + \frac{2}{3}\, x + \frac{1}{4}\, x^2 + \frac{1}{15}\, x^3 + \cdots\Big].$$

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Find the solution y near the regular-singular point $x_0 = 0$ of

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One can also check that the solution y_- is

$$y_- = a_2\, x\Big[x^2 + \frac{2}{3}\, x^3 + \frac{1}{4}\, x^4 + \frac{1}{15}\, x^5 + \cdots\Big].$$

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Notice:

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Notice:

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Solution: The solutions y_+ and y_- are not linearly independent.

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Remark: It can be shown the following result:
If the roots of the Euler characteristic polynomial r_+, r_- differ by an integer, then the second solution y_-, the solution corresponding to the smaller root, is not given by the method above.

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This solution involves logarithmic terms.
We do not study this type of solutions in these notes. $\triangleleft$