Second order linear homogeneous ODE (Sect. 3.4).

- Review: On solutions of $y'' + a_1\,y' + a_0\,y = 0.$
- Repeated roots as a limit case.
- Main result for repeated roots.
- Reduction of the order method:
  - Constant coefficients equations.
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Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Summary:

Given constants $a_1,\ a_0 \in \mathbb{R}$, consider the differential equation

$$y'' + a_1 y' + a_0 y = 0$$

with characteristic polynomial having roots

$$r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

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Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Question:

Consider the case (3), with $a_1^2 - 4a_0 = 0$, that is, $a_0 = \dfrac{a_1^2}{4}$.

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Consider the case (3), with $a_1^2 - 4a_0 = 0$, that is, $a_0 = \dfrac{a_1^2}{4}$.

- Does the equation

$$y'' + a_1 y' + \frac{a_1^2}{4}\, y = 0$$

  have two linearly independent solutions?

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

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Consider the case (3), with $a_1^2 - 4a_0 = 0$, that is, $a_0 = \dfrac{a_1^2}{4}$.

- Does the equation

$$y'' + a_1 y' + \frac{a_1^2}{4}\, y = 0$$

have two linearly independent solutions?

- Or, every solution to the equation above is proportional to

$$y_1(t) = e^{-\frac{a_1}{2}\, t}.$$

Second order linear homogeneous ODE (Sect. 3.4).

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- **Repeated roots as a limit case.**
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Repeated roots as a limit case.

Remark:

- Case (3), where $4a_0 - a_1^2 = 0$ can be obtained as the limit $\beta \to 0$ in case (2).

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- Is $y_2(t) = t\, y_1(t)$ solution of the differential equation? Introducing y_2 in the differential equation one obtains: Yes.
- Since y_2 is not proportional to y_1, the functions y_1, y_2 are a fundamental set for the differential equation in case (3).

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Main result for repeated roots.

Theorem

If a_1, $a_0 \in R$ satisfy that $a_1^2 = 4a_0$, then the functions

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The Theorem above implies that the general solution is

$$y(t) = (c_1 + c_2 t)\,e^{-t/3}. \qquad \triangleleft$$

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Proof case $a_1^2 - 4a_0 = 0$:

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The hypothesis $a_1^2 = 4a_0$ implies $r_+ = r_- = -a_1/2$.

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So, the solution r_+ of the characteristic equation satisfies both

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A second solution y_2 not proportional to y_1 can be found as follows: (D'Alembert $\sim$ 1750.)

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It is clear that $y_1(t) = e^{r_+ t}$ is solutions of the differential equation.

A second solution y_2 not proportional to y_1 can be found as follows: (D'Alembert $\sim$ 1750.)

Express: $y_2(t) = v(t)\, y_1(t)$, and find the equation that function v satisfies from the condition $y_2'' + a_1 y_2' + a_0 y_2 = 0$.

Reduction of the order method: Constant coefficients.

Recall: $y_2 = v y_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$.

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Recall: $y_2 = v y_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. So, $y_2 = v e^{r_+ t}$ and

$$y_2' = v' e^{r_+ t} + r_+ v e^{r_+ t}, \qquad y_2'' = v'' e^{r_+ t} + 2 r_+ v' e^{r_+ t} + r_+^2 v e^{r_+ t}.$$

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Introducing this information into the differential equation

$$\left[v'' + 2r_+ v' + r_+^2 v\right] e^{r_+ t} + a_1 \left[v' + r_+ v\right] e^{r_+ t} + a_0 v\, e^{r_+ t} = 0.$$

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$$v'' + \left(2r_+ + a_1\right) v' + \left(r_+^2 + a_1 r_+ + a_0\right) v = 0$$

Reduction of the order method: Constant coefficients.

Recall: $y_2 = v y_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. So, $y_2 = v e^{r_+ t}$ and

$$y_2' = v' e^{r_+ t} + r_+ v e^{r_+ t}, \qquad y_2'' = v'' e^{r_+ t} + 2 r_+ v' e^{r_+ t} + r_+^2 v e^{r_+ t}.$$

Introducing this information into the differential equation

$$\left[v'' + 2 r_+ v' + r_+^2 v\right] e^{r_+ t} + a_1 \left[v' + r_+ v\right] e^{r_+ t} + a_0 v \, e^{r_+ t} = 0.$$

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Recall that r_+ satisfies: $r_+^2 + a_1 r_+ + a_0 = 0$ and $2 r_+ + a_1 = 0$.

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$$v'' = 0$$

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Introducing this information into the differential equation

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$$v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t)$$

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Introducing this information into the differential equation

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$$v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t) \quad \Rightarrow \quad y_2 = (c_1 + c_2 t)\, e^{r_+ t}.$$

Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t)\, e^{r_+ t}$.

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If $c_2 = 0$, then $y_2 = c_1 e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly dependent functions.

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Simplest choice: $c_1 = 0$ and $c_2 = 1$.

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$$y_1(t) = e^{r_+ t}, \qquad y_2(t) = t\, e^{r_+ t} \qquad \square$$

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The general solution to the differential equation is

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Reduction of the order method: Constant coefficients.

Example

Find the solution to the initial value problem

$$9y'' + 6y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = \frac{5}{3}.$$

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The initial conditions imply that

$$\left.\begin{aligned} 1 &= y(0) = c_1, \\ \frac{5}{3} &= y'(0) = -\frac{c_1}{3} + c_2 \end{aligned}\right\} \quad \Rightarrow \quad c_1 = 1, \qquad c_2 = 2.$$

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We conclude that $y(t) = (1 + 2t)\, e^{-t/3}$. $\triangleleft$

Second order linear homogeneous ODE (Sect. 3.4).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Repeated roots as a limit case.
- Main result for repeated roots.
- **Reduction of the order method:**
  - Constant coefficients equations.
  - **Variable coefficients equations.**

Reduction of the order method: Variable coefficients.

Remark: The same idea used to prove the constant coefficients Theorem above can be used in variable coefficients equations.

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Theorem

Given continuous functions $p,\, q : (t_1, t_2) \to \mathbb{R}$, *let* $y_1 : (t_1, t_2) \to \mathbb{R}$ *be a solution of*

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

If the function $v : (t_1, t_2) \to \mathbb{R}$ *is solution of*

$$y_1(t)\, v'' + \big[2y'(t) + p(t) y_1(t)\big]\, v' = 0. \tag{1}$$

then the functions y_1 *and* $y_2 = v\, y_1$ *are fundamental solutions to the differential equation above.*

Reduction of the order method: Variable coefficients.

Remark: The same idea used to prove the constant coefficients Theorem above can be used in variable coefficients equations.

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Remark: The reason for the name Reduction of order method is that the function v does not appear in Eq. (1). This is a first order equation in v'.

Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2t y' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

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Example

Find a fundamental set of solutions to

$$t^2 y'' + 2t y' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t)\, y_1(t)$.

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Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2t y_2' - 2y_2 = 0$. We need to compute

$$y_2 = v\, t,$$

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So, the equation for v is given by

$$t^2\big(t\, v'' + 2v'\big) + 2t\big(t\, v' + v\big) - 2t\, v = 0$$

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Find a fundamental set of solutions to

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$$t^3\,v'' + (4t^2)\,v' = 0$$

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Integrating w we obtain v, that is, $v = c_2 t^{-3} + c_3$, with $c_2, c_3 \in \mathbb{R}$.

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Recalling that $y_2 = t\, v$

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This is a first order equation for $w = v'$, given by $w' + \frac{4}{t} w = 0$, so

$$\frac{w'}{w} = -\frac{4}{t} \;\Rightarrow\; \ln(w) = -4\ln(t) + c_0 \;\Rightarrow\; w(t) = c_1 t^{-4}, \; c_1 \in \mathbb{R}.$$

Integrating w we obtain v, that is, $v = c_2 t^{-3} + c_3$, with $c_2, c_3 \in \mathbb{R}$.
Recalling that $y_2 = t\,v$ we then conclude that $y_2 = c_2 t^{-2} + c_3 t$.

Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2t y' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

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Recalling that $y_2 = t\,v$ we then conclude that $y_2 = c_2 t^{-2} + c_3 t$.
Choosing $c_2 = 1$ and $c_3 = 0$ we obtain the fundamental solutions

$y_1(t) = t$ and $y_2(t) = \dfrac{1}{t^2}$. $\triangleleft$

Reduction of the order method: Variable coefficients.

Proof of the Theorem: The choice of $y_2 = v y_1$ implies

$$y_2' = v'\, y_1 + v\, y_1', \qquad y_2'' = v''\, y_1 + 2v'\, y_1' + v\, y_1''.$$

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This information introduced into the differential equation says that

$$\big(v''\, y_1 + 2v'\, y_1' + v\, y_1''\big) + p\,\big(v'\, y_1 + v\, y_1'\big) + qv\, y_1 = 0$$

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Then, the equation for v is given by Eq. (1), that is,

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Reduction of the order method: Variable coefficients.

Proof: Recall $y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0.$

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Proof: Recall $y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0$. We now need to show that y_1 and $y_2 = vy_1$ are linearly independent.

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$$W_{y_1 y_2} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & (v' y_1 + vy_1') \end{vmatrix}$$

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Non-homogeneous equations (Sect. 3.5).

- We study: $y'' + a_1\, y' + a_0\, y = b(t)$.
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
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Operator notation and preliminary results.

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The function L acting on a function y is called an operator.

Operator notation and preliminary results.

Remark: The operator L is a linear function of y.

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Theorem

For every continuously differentiable functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ and every c_1, $c_2 \in \mathbb{R}$ holds that

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$

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$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2). \qquad \square$$

Operator notation and preliminary results.

Theorem

Given functions p, q, f, *let* $L(y) = y'' + p(t)\,y' + q(t)\,y$.
If the functions y_1 *and* y_2 *are fundamental solutions of the homogeneous equation*

$$L(y) = 0,$$

and y_p *is any solution of the non-homogeneous equation*

$$L(y_p) = f, \tag{2}$$

then any other solution y *of the non-homogeneous equation above is given by*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \tag{3}$$

where c_1, $c_2 \in \mathbb{R}$.

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Notation: The expression for y in Eq. (3) is called the general solution of the non-homogeneous Eq. (2).

Operator notation and preliminary results.

Theorem

Given functions p, q, let $L(y) = y'' + p(t)\,y' + q(t)\,y$.
If the function f can be written as $f(t) = f_1(t) + \cdots + f_n(t)$, with $n \geqslant 1$, and if there exist functions $y_{p_1}, \cdots, y_{p_n}$ such that

$$L(y_{p_i}) = f_i, \qquad i = 1, \cdots, n,$$

then the function $y_p = y_{p_1} + \cdots + y_{p_n}$ satisfies the non-homogeneous equation

$$L(y_p) = f.$$

Non-homogeneous equations (Sect. 3.5).

- We study: $y'' + a_1\,y' + a_0\,y = b(t)$.
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- **Summary of the undetermined coefficients method.**
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Summary of the undetermined coefficients method.

Problem: Given a constant coefficients linear operator $L(y) = y'' + a_1 y' + a_0 y$, with a_1, $a_2 \in \mathbb{R}$, find every solution of the non-homogeneous differential equation

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Remarks:

- The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.

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Problem: Given a constant coefficients linear operator $L(y) = y'' + a_1 y' + a_0 y$, with $a_1,\ a_2 \in \mathbb{R}$, find every solution of the non-homogeneous differential equation

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Remarks:

- The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.
- It consists in guessing the solution y_p of the non-homogeneous equation

$$L(y_p) = f,$$

  for particularly simple source functions f.

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(1) Find the general solution of the homogeneous equation $L(y_h) = 0$.

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(1) Find the general solution of the homogeneous equation $L(y_h) = 0$.

(2) If f has the form $f = f_1 + \cdots + f_n$, with $n \geqslant 1$, then look for solutions y_{p_i}, with $i = 1, \cdots, n$ to the equations

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Once the functions y_{p_i} are found, then construct

$$y_p = y_{p_1} + \cdots + y_{p_n}.$$

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Once the functions y_{p_i} are found, then construct

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(3) Given the source functions f_i, guess the solutions functions y_{p_i} following the Table below.

Summary of the undetermined coefficients method.

Summary (cont.):

$f_i(t)$ (K, m, a, b, given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$K\sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at}\cos(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$KKe^{at}\sin(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$Kt^m\cos(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$
$Kt^m\sin(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$

Summary of the undetermined coefficients method.

Summary (cont.):

(4) If any guessed function y_{p_i} satisfies the homogeneous equation $L(y_{p_i}) = 0$, then change the guess to the function

$$t^s y_{p_i}, \quad \text{with} \quad s \geqslant 1,$$

and s sufficiently large such that $L(t^s y_{p_i}) \neq 0$.

Summary of the undetermined coefficients method.

Summary (cont.):

(4) If any guessed function y_{p_i} satisfies the homogeneous equation $L(y_{p_i}) = 0$, then change the guess to the function

$$t^s y_{p_i}, \quad \text{with} \quad s \geqslant 1,$$

and s sufficiently large such that $L(t^s y_{p_i}) \neq 0$.

(5) Impose the equation $L(y_{p_i}) = f_i$ to find the undetermined constants $k_1, \cdots, k_m$, for the appropriate m, given in the table above.

Summary of the undetermined coefficients method.

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(4) If any guessed function y_{p_i} satisfies the homogeneous equation $L(y_{p_i}) = 0$, then change the guess to the function

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(6) The general solution to the original differential equation $L(y) = f$ is then given by

$$y(t) = y_h(t) + y_{p_1} + \cdots + y_{p_n}.$$

Non-homogeneous equations (Sect. 3.5).

- We study: $y'' + a_1\, y' + a_0\, y = b(t)$.
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- **Using the method in few examples.**
- The guessing solution table.

Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution: Notice: $L(y) = y'' - 3y' - 4y$ and $f(t) = 3e^{2t}$.

Using the method in few examples.

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(1) Find all solutions y_h to the homogeneous equation $L(y_h) = 0$.

Using the method in few examples.

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The characteristic equation is

$$r^2 - 3r - 4 = 0$$

Using the method in few examples.

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Using the method in few examples.

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$$y_h(t) = c_1 \, e^{4t} + c_2 \, e^{-t}.$$

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Find all solutions to the non-homogeneous equation

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$$y_h(t) = c_1\, e^{4t} + c_2\, e^{-t}.$$

(2) Trivial in our case. The source function $f(t) = 3e^{2t}$ cannot be simplified into a sum of simpler functions.

Using the method in few examples.

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(2) Trivial in our case. The source function $f(t) = 3e^{2t}$ cannot be simplified into a sum of simpler functions.

(3) Table says: For $f(t) = 3e^{2t}$ guess $y_p(t) = k\, e^{2t}$

Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution: Recall: $y_p(t) = k\, e^{2t}$. We need to find k.

Using the method in few examples.

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$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution: Recall: $y_p(t) = k\,e^{2t}$. We need to find k.

(4) Trivial here, since $L(y_p) \neq 0$, we do not modify our guess.

Using the method in few examples.

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(5) Introduce y_p into $L(y_p) = f$ and find k.

Using the method in few examples.

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$$(2^2 - 6 - 4)ke^{2t} = 3e^{2t}$$

Using the method in few examples.

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(5) Introduce y_p into $L(y_p) = f$ and find k.

$$(2^2 - 6 - 4)ke^{2t} = 3e^{2t} \quad \Rightarrow \quad -6k = 3$$

Using the method in few examples.

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Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

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Using the method in few examples.

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We have obtained that $y_p(t) = -\frac{1}{2}\, e^{2t}$.

Using the method in few examples.

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(6) The general solution to the inhomogeneous equation is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}\,e^{2t}.$$

$\triangleleft$

Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{4t}.$$

Using the method in few examples.

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Following the table we guess y_p as $y_p = k\, e^{4t}$.

Using the method in few examples.

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However, this guess satisfies $L(y_p) = 0$.

Using the method in few examples.

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However, this guess satisfies $L(y_p) = 0$.

So we modify the guess to $y_p = kt\, e^{4t}$.

Using the method in few examples.

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Introduce the guess into $L(y_p) = f$.

Using the method in few examples.

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Introduce the guess into $L(y_p) = f$. We need to compute

$$y_p' = k\, e^{4t} + 4kt\, e^{4t}, \qquad y_p'' = 8k\, e^{4t} + 16kt\, e^{4t}.$$

Using the method in few examples.

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Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{4t}.$$

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$$\big[(8+16t) - 3(1+4t) - 4t\big]\,k = 3 \quad \Rightarrow \quad \big[5 + (16-12-4)\,t\big]\,k = 3$$

Using the method in few examples.

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We obtain that $k = \dfrac{3}{5}$.

Using the method in few examples.

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We obtain that $k = \dfrac{3}{5}$. Therefore, $y_p(t) = \dfrac{3}{5}\, t\, e^{4t}$,

Using the method in few examples.

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We obtain that $k = \frac{3}{5}$. Therefore, $y_p(t) = \frac{3}{5}\, t\, e^{4t}$, and

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{3}{5}\, t\, e^{4t}. \qquad \triangleleft$$

Using the method in few examples.

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Using the method in few examples.

Example

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$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: We know that the general solution to homogeneous equation is $y(t) = c_1 e^{4t} + c_2 e^{-t}$.

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Using the method in few examples.

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Following the table: Since $f = 2\sin(t)$, then we guess

$$y_p = k_1 \sin(t) + k_2 \cos(t).$$

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Solution: We know that the general solution to homogeneous equation is $y(t) = c_1 e^{4t} + c_2 e^{-t}$.

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$$y_p = k_1 \sin(t) + k_2 \cos(t).$$

This guess satisfies $L(y_p) \neq 0$.

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This guess satisfies $L(y_p) \neq 0$.

Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

Using the method in few examples.

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Following the table: Since $f = 2\sin(t)$, then we guess

$$y_p = k_1 \sin(t) + k_2 \cos(t).$$

This guess satisfies $L(y_p) \neq 0$.

Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$

$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

Using the method in few examples.

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$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

$$(-5k_1 + 3k_2)\sin(t) + (-3k_1 - 5k_2)\cos(t) = 2\sin(t).$$

Using the method in few examples.

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Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

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This equation holds for all $t \in \mathbb{R}$. In particular, at $t = \dfrac{\pi}{2}$, $t = 0$.

$$\begin{aligned} -5k_1 + 3k_2 &= 2, \\ -3k_1 - 5k_2 &= 0, \end{aligned}$$

Using the method in few examples.

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$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$
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$$\left.\begin{aligned} -5k_1 + 3k_2 &= 2, \\ -3k_1 - 5k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} k_1 &= -\frac{5}{17}, \\ k_2 &= \frac{3}{17}. \end{aligned}\right.$$

Using the method in few examples.

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Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall: $k_1 = -\frac{5}{17}$ and $k_2 = \frac{3}{17}$.

Using the method in few examples.

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Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall: $k_1 = -\dfrac{5}{17}$ and $k_2 = \dfrac{3}{17}$.

So the particular solution to the inhomogeneous equation is

$$y_p(t) = \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right].$$

Using the method in few examples.

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So the particular solution to the inhomogeneous equation is

$$y_p(t) = \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right].$$

The general solution is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right]. \qquad \triangleleft$$

Using the method in few examples.

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t).$$

Using the method in few examples.

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t).$$

Solution: We know that the general solution y is given by

$$y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t),$$

Using the method in few examples.

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Using the method in few examples.

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where $y_h(t) = c_1 e^{4t} + c_2 e^{2t}$, $L(y_{p_1}) = 3e^{2t}$, and $L(y_{p_2}) = 2\sin(t)$.

Using the method in few examples.

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Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t).$$

Solution: We know that the general solution y is given by

$$y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t),$$

where $y_h(t) = c_1 e^{4t} + c_2 e^{2t}$, $L(y_{p_1}) = 3e^{2t}$, and $L(y_{p_2}) = 2\sin(t)$.
We have just found out that

$$y_p(t) = -\frac{1}{2}\,e^{2t}, \qquad y_{p_2}(t) = \frac{1}{17}\,\big[-5\sin(t) + 3\cos(t)\big].$$

Using the method in few examples.

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t).$$

Solution: We know that the general solution y is given by

$$y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t),$$

where $y_h(t) = c_1 e^{4t} + c_2 e^{2t}$, $L(y_{p_1}) = 3e^{2t}$, and $L(y_{p_2}) = 2\sin(t)$. We have just found out that

$$y_p(t) = -\frac{1}{2}\, e^{2t}, \qquad y_{p_2}(t) = \frac{1}{17}\,\big[-5\sin(t) + 3\cos(t)\big].$$

We conclude that

$$y(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{1}{2}\, e^{2t} + \frac{1}{17}\,\big[-5\sin(t) + 3\cos(t)\big]. \qquad \triangleleft$$

Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$,

Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$

Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess
$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$
- For $y'' - 3y' - 4y = 2t^2\, e^{3t}$,

Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \big[k_1 \sin(t) + k_2 \cos(t)\big]\, e^{2t}.$$

- For $y'' - 3y' - 4y = 2t^2\, e^{3t}$, guess

$$y_p(t) = \big(k_0 + k_1 t + k_2 t^2\big)\, e^{3t}.$$

Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \left[k_1\sin(t) + k_2\cos(t)\right] e^{2t}.$$

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Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \big[k_1 \sin(t) + k_2 \cos(t)\big]\, e^{2t}.$$

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- For $y'' - 3y' - 4y = 3t\,\sin(t)$, guess

$$y_p(t) = (1 + k_1 t)\,\big[k_2 \sin(t) + k_3 \cos(t)\big].$$

Non-homogeneous equations (Sect. 3.5).

- We study: $y'' + a_1\, y' + a_0\, y = b(t)$.
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
- **The guessing solution table.**

The guessing solution table.

Guessing Solution Table.

$f_i(t)$ (K, m, a, b, given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$K\sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at}\cos(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$KKe^{at}\sin(bt)$	$e^{at}\big[k_1 \cos(bt) + k_2 \sin(bt)\big]$
$Kt^m\cos(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$
$Kt^m\sin(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1 \cos(bt) + a_2 \sin(bt)\big]$

Non-homogeneous equations (Sect. 3.6).

- We study: $y'' + p(t)\,y' + q(t)\,y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- Using the method in another example.

Method of variation of parameters.

Remarks:

- This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function,

$$y'' + p(t)\,y' + q(t)\,y = f(t).$$

Method of variation of parameters.

Remarks:

- This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function,

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- The variation of parameter method can be applied to more general equations than the undetermined coefficients method.

Method of variation of parameters.

Remarks:

- This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function,

$$y'' + p(t)\,y' + q(t)\,y = f(t).$$

- The variation of parameter method can be applied to more general equations than the undetermined coefficients method.
- The variation of parameter method usually takes more time to implement than the simpler method of undetermined coefficients.

Method of variation of parameters.

Theorem (Variation of parameters)

Let p, q, $f : (t_1, t_2) \to \mathbb{R}$ be continuous functions, let y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ be linearly independent solutions to the homogeneous equation

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

and let $W_{y_1 y_2}$ be the Wronskian of y_1 and y_2. If the functions u_1 and u_2 are defined by

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt,$$

then the function $y_p = u_1 y_1 + u_2 y_2$ is a particular solution to the non-homogeneous equation

$$y'' + p(t)\, y' + q(t)\, y = f(t).$$

Non-homogeneous equations (Sect. 3.6).

- We study: $y'' + p(t)\, y' + q(t)\, y = f(t)$.
- Method of variation of parameters.
- **Using the method in an example.**
- The proof of the variation of parameter method.
- Using the method in another example.

Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^t.$$

Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^{t}.$$

Solution:

First: Find fundamental solutions to the homogeneous equation.

Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^{t}.$$

Solution:

First: Find fundamental solutions to the homogeneous equation.
The characteristic equation is

$$r^2 - 5r + 6 = 0$$

Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^{t}.$$

Solution:

First: Find fundamental solutions to the homogeneous equation.
The characteristic equation is

$$r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r = \frac{1}{2}\big(5 \pm \sqrt{25 - 24}\big)$$

Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^t.$$

Solution:

First: Find fundamental solutions to the homogeneous equation.

The characteristic equation is

$$r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r = \frac{1}{2}\left(5 \pm \sqrt{25 - 24}\right) \quad \Rightarrow \quad \begin{cases} r_1 = 3, \\ r_2 = 2. \end{cases}$$

Using the method in an example.

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Hence, $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$.

Using the method in an example.

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Hence, $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$. Compute their Wronskian,

$$W_{y_1 y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t})$$

Using the method in an example.

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Find the general solution of the inhomogeneous equation

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First: Find fundamental solutions to the homogeneous equation.
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$$W_{y_1 y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -e^{5t}.$$

Using the method in an example.

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Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^{t}.$$

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The characteristic equation is

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Second: We compute the functions u_1 and u_2.

Using the method in an example.

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First: Find fundamental solutions to the homogeneous equation. The characteristic equation is

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Hence, $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$. Compute their Wronskian,

$$W_{y_1 y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -e^{5t}.$$

Second: We compute the functions u_1 and u_2. By definition,

$$u_1' = -\frac{y_2 f}{W_{y_1 y_2}}, \qquad u_2' = \frac{y_1 f}{W_{y_1 y_2}}.$$

Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^t.$$

Solution: Recall: $y_1(t) = e^{3t}$, $y_2(t) = e^{2t}$, $W_{y_1 y_2}(t) = -e^{5t}$, and

$$u_1' = -\frac{y_2 f}{W_{y_1 y_2}}, \qquad u_2' = \frac{y_1 f}{W_{y_1 y_2}}.$$

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Third: The particular solution is

$$y_p = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t})$$

Using the method in an example.

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$$y_p = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t}) \quad \Rightarrow \quad y_p = e^t.$$

The general solution is $y(t) = c_1 e^{3t} + c_2 e^{2t} + e^t$, $c_1, c_2 \in \mathbb{R}$. $\triangleleft$

Non-homogeneous equations (Sect. 3.6).

- We study: $y'' + p(t)\,y' + q(t)\,y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- **The proof of the variation of parameter method.**
- Using the method in another example.

The proof of the variation of parameter method.

Proof: Denote $L(y) = y'' + p(t)\,y' + q(t)\,y$.

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We hope that the equation for u_1 and u_2 will be simpler than the original equation for y_p,

The proof of the variation of parameter method.

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First idea: Propose that y_p is given by $y_p = u_1 y_1 + u_2 y_2$.

We hope that the equation for u_1 and u_2 will be simpler than the original equation for y_p, since y_1 and y_2 are solutions to the homogeneous equation.

The proof of the variation of parameter method.

Proof: Denote $L(y) = y'' + p(t)\, y' + q(t)\, y$.

We need to find y_p solution of $L(y_p) = f$.

We know y_1 and y_2 solutions of $L(y_1) = 0$ and $L(y_2) = 0$.

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First idea: Propose that y_p is given by $y_p = u_1 y_1 + u_2 y_2$.

We hope that the equation for u_1 and u_2 will be simpler than the original equation for y_p, since y_1 and y_2 are solutions to the homogeneous equation. Compute:

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2',$$

The proof of the variation of parameter method.

Proof: Denote $L(y) = y'' + p(t)\,y' + q(t)\,y$.

We need to find y_p solution of $L(y_p) = f$.

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Idea: The reduction of order method: Find y_2 proposing $y_2 = uy_1$.

First idea: Propose that y_p is given by $y_p = u_1 y_1 + u_2 y_2$.

We hope that the equation for u_1 and u_2 will be simpler than the original equation for y_p, since y_1 and y_2 are solutions to the homogeneous equation. Compute:

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2',$$

$$y_p'' = u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''.$$

The proof of the variation of parameter method.

Proof: Then $L(y_p) = f$ is given by

$$\big[u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''\big]$$

$$p(t)\big[u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'\big] + q(t)\big[u_1y_1 + u_2y_2\big] = f(t).$$

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$$\left[u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''\right]$$

$$p(t)\left[u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'\right] + q(t)\left[u_1 y_1 + u_2 y_2\right] = f(t).$$

$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2 y_2') + p\,(u_1' y_1 + u_2' y_2)$$

$$+ u_1\,(y_1'' + p\,y_1' + q\,y_1) + u_2\,(y_2'' + p\,y_2' + q\,y_2) = f$$

The proof of the variation of parameter method.

Proof: Then $L(y_p) = f$ is given by

$$\left[u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''\right]$$

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Recall: $y_1'' + p\,y_1' + q\,y_1 = 0$ and $y_2'' + p\,y_2' + q\,y_2 = 0$.

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$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2' y_2') + p\,(u_1' y_1 + u_2' y_2) = f$$

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$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2' y_2') + p\,(u_1' y_1 + u_2' y_2) = f$$

Second idea: Look for u_1 and u_2 that satisfy the extra equation

$$u_1' y_1 + u_2' y_2 = 0.$$

The proof of the variation of parameter method.

Proof: Recall: $u_1' y_1 + u_2' y_2 = 0$ and

$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2' y_2') + p\,(u_1' y_1 + u_2' y_2) = f.$$

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These two equations imply that $L(y_p) = f$ is

$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2' y_2') = f.$$

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The proof of the variation of parameter method.

Proof: Recall: $u_1' y_1 + u_2' y_2 = 0$ and

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$$u_1'' y_1 + u_2'' y_2 + (u_1' y_1' + u_2' y_2') = 0.$$

The proof of the variation of parameter method.

Proof: Recall: $u_1' y_1 + u_2' y_2 = 0$ and

$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2' y_2') + p\,(u_1' y_1 + u_2' y_2) = f.$$

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This information in $L(y_p) = f$ implies

$$u_1' y_1' + u_2' y_2' = f.$$

The proof of the variation of parameter method.

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This information in $L(y_p) = f$ implies

$$u_1' y_1' + u_2' y_2' = f.$$

Summary: If u_1 and u_2 satisfy $u_1' y_1 + u_2' y_2 = 0$ and $u_1' y_1' + u_2' y_2' = f$, then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

The proof of the variation of parameter method.

Proof: Summary: If u_1 and u_2 satisfy $u_1' y_1 + u_2' y_2 = 0$ and $u_1' y_1' + u_2' y_2' = f$, then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

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The equations above are simple to solve for u_1 and u_2,

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$,

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Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$,

$$u_1' = -\frac{y_2 f}{W_{y_1 y_2}} \quad \Rightarrow \quad u_2' = \frac{y_1 f}{W_{y_1 y_2}}.$$

Integrating in the variable t we obtain

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt,$$

This establishes the Theorem. $\square$

Non-homogeneous equations (Sect. 3.6).

- We study: $y'' + p(t)\, y' + q(t)\, y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- **Using the method in another example.**

Using the method in another example.

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Using the method in another example.

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Solution: First, write the equation in the form of the Theorem. That is, divide the whole equation by t^2,

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Using the method in another example.

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We know that $y_1 = t^2$ and $y_2 = 1/t$. Their Wronskian is

$$W_{y_1 y_2}(t) = (t^2)\Big(\frac{-1}{t^2}\Big) - (2t)\Big(\frac{1}{t}\Big)$$

Using the method in another example.

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Using the method in another example.

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knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: $y_1 = t^2$, $y_2 = 1/t$, $f(t) = 3 - \dfrac{1}{t^2}$, $W_{y_1 y_2}(t) = -3$.

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Solution: $y_1 = t^2$, $y_2 = 1/t$, $f(t) = 3 - \dfrac{1}{t^2}$, $W_{y_1 y_2}(t) = -3$.

We now compute y_1 and u_2,

$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3}$$

Using the method in another example.

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$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = \frac{1}{t} - \frac{1}{3}\,t^{-3}$$

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$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = \frac{1}{t} - \frac{1}{3}\,t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6}t^{-2},$$

Using the method in another example.

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knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: $y_1 = t^2, \;\; y_2 = 1/t, \;\; f(t) = 3 - \dfrac{1}{t^2}, \; W_{y_1 y_2}(t) = -3.$

We now compute y_1 and u_2,

$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = \frac{1}{t} - \frac{1}{3}\,t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6}t^{-2},$$

$$u_2' = (t^2)\left(3 - \frac{1}{t^2}\right)\frac{1}{-3}$$

Using the method in another example.

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knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: $y_1 = t^2$, $y_2 = 1/t$, $f(t) = 3 - \frac{1}{t^2}$, $W_{y_1 y_2}(t) = -3$.

We now compute y_1 and u_2,

$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = \frac{1}{t} - \frac{1}{3}\,t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6}t^{-2},$$

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Using the method in another example.

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Using the method in another example.

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Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: The particular solution $\tilde{y}_p = u_1 y_1 + u_2 y_2$ is

$$\tilde{y}_p = \left[\ln(t) + \frac{1}{6} t^{-2}\right](t^2) + \frac{1}{3}(-t^3 + t)(t^{-1})$$

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Using the method in another example.

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$$\tilde{y}_p = \Big[\ln(t) + \frac{1}{6} t^{-2}\Big](t^2) + \frac{1}{3}(-t^3 + t)(t^{-1})$$

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Using the method in another example.

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

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A simpler expression is $y_p = t^2 \ln(t) + \frac{1}{2}$. $\triangleleft$

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$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f.$$

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$$u_2' = -t^3\, u_1' \;\Rightarrow\; 2t\, u_1' + t\, u_1' = 3 - \frac{1}{t^2} \;\Rightarrow\; \begin{cases} u_1' = \dfrac{1}{t} - \dfrac{1}{3t^3} \\[2ex] u_2' = -t^2 + \dfrac{1}{3}. \end{cases}$$