Autonomous systems (Sect. 2.5).

- Definition and examples.
- Qualitative analysis of the solutions.
- Equilibrium solutions and stability.
- Population growth equation.

Definition and examples

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Without a computer it is difficult to graph the solution.

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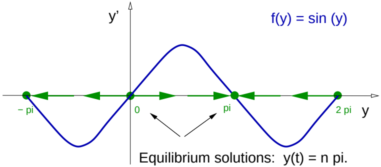

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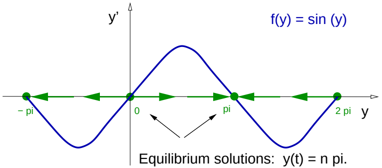

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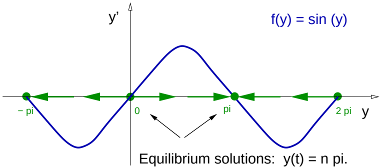

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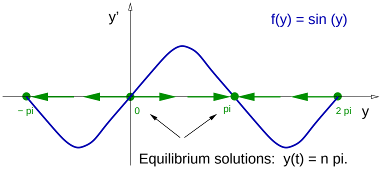

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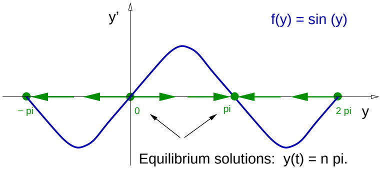

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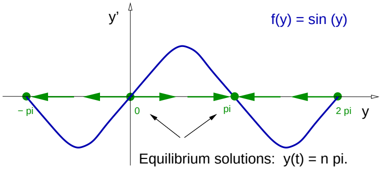

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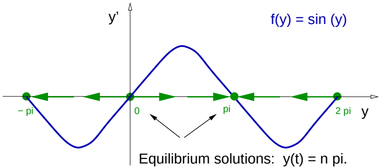

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They are called equilibrium solutions.

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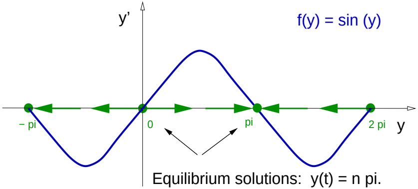

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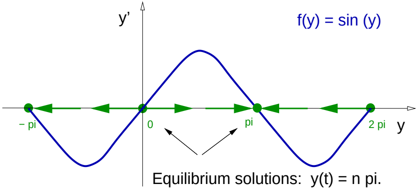

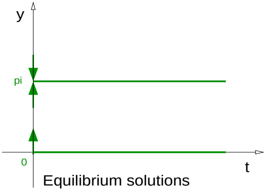

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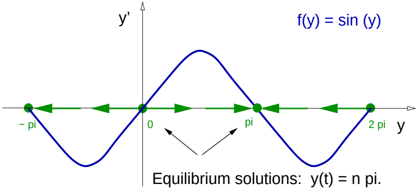

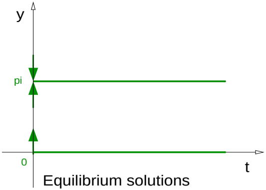

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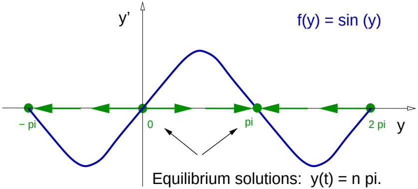

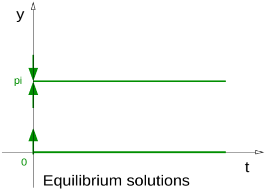

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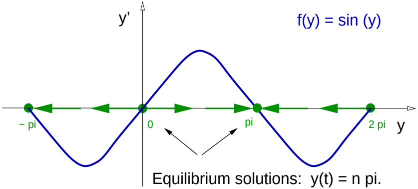

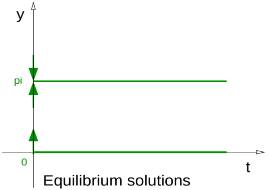

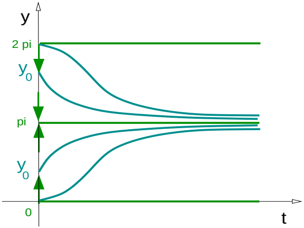

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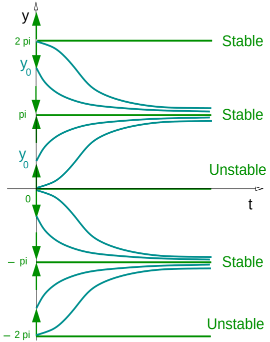
- Population growth equation.

Equilibrium solutions and stability



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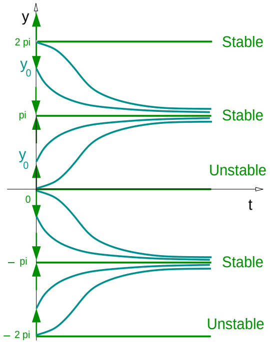

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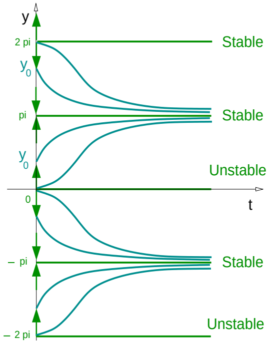

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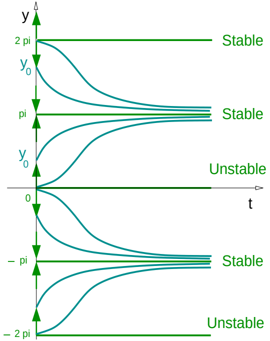

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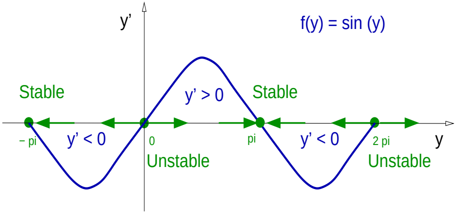

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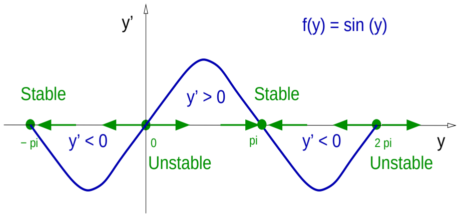

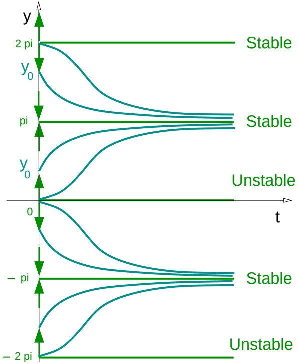

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Population growth equation (Logistic equation)

Example

Sketch a qualitative graph of solutions for different initial data conditions $y(0) = y_0$ to the population growth equation $y' = r\left(1 - \frac{y}{K}\right) y$, where r and K are given positive constants.

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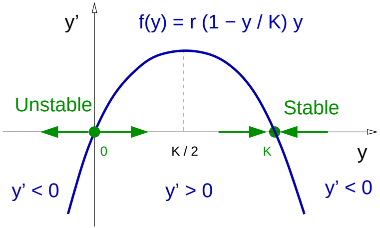

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Sketch a qualitative graph of solutions for different initial data conditions $y(0) = y_0$ to the population growth equation $y' = r\left(1 - \frac{y}{K}\right) y$, where r and K are given positive constants.

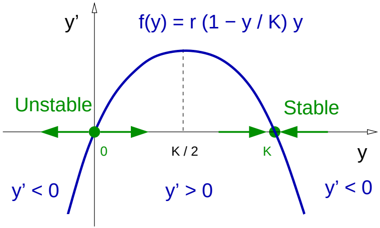
Solution:

(1) Plot the function

$$f(y) = r\left(1 - \frac{y}{K}\right)y.$$

(2) Find the zeros of f.

$$y_0 = 0, \quad y_0 = K.$$



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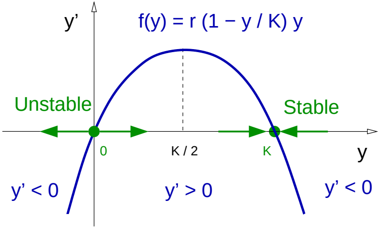

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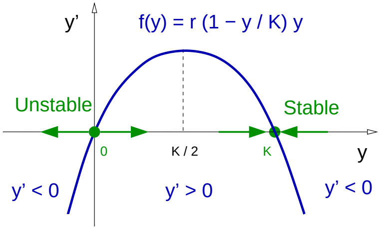

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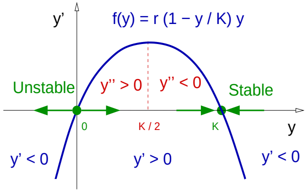

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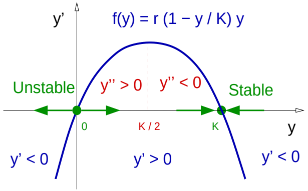

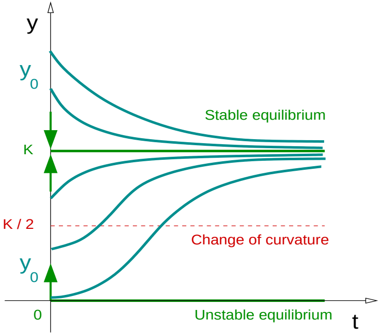

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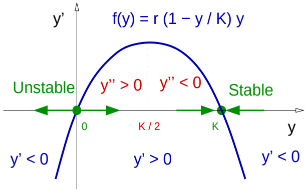

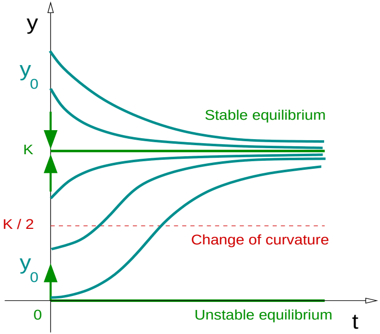

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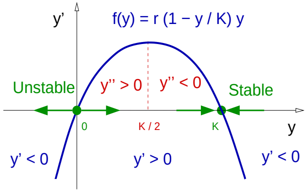

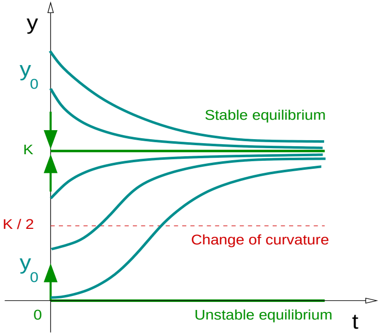

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Remark: The curvature of the solution y depends on $f'(y)\, f(y)$.

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Theorem

If the function y is a solution of the autonomous system $y' = f(y)$, then the graph of y has positive curvature iff $f'(y)\, f(y) > 0$, and negative curvature iff $f'(y)\, f(y) < 0$.

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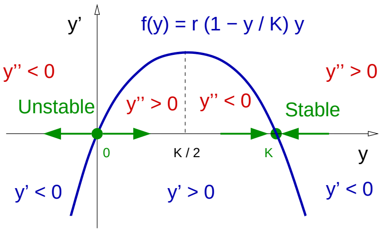

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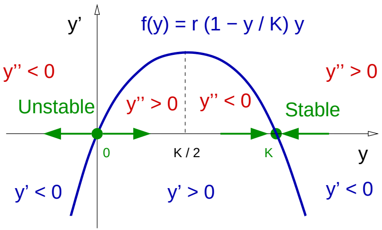

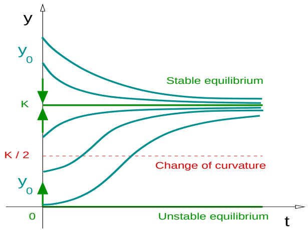

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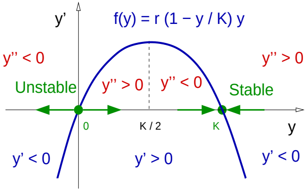

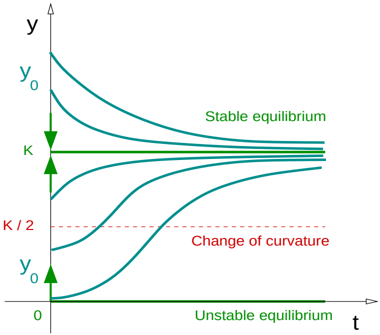

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Exact equations (Sect. 2.6).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.

Exact differential equations.

Definition

Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ and continuously differentiable functions $M, N : R \to \mathbb{R}$, denoted as $(t, u) \mapsto M(t, u)$ and $(t, u) \mapsto N(t, u)$, the differential equation in the unknown function $y : (t_1, t_2) \to \mathbb{R}$ given by

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Recall: we use the notation: $\partial_t N = \dfrac{\partial N}{\partial t}$, and $\partial_u M = \dfrac{\partial M}{\partial u}$.

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Show whether the differential equation below is exact,

$$2ty(t)\,y'(t) + 2t + y^2(t) = 0.$$

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Remark: The ODE above is not separable and non-linear.

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$$N(t, u) = \sin(t) + t^2 e^u - 1$$

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$$\left[\sin(t) + t^2 e^{y(t)} - 1\right] y'(t) + \left[y(t)\cos(t) + 2te^{y(t)}\right] = 0,$$

we can see that

$$N(t,u) = \sin(t) + t^2 e^u - 1 \quad \Rightarrow \quad \partial_t N(t,u) = \cos(t) + 2te^u,$$

Exact differential equations.

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Show whether the differential equation below is exact,

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This implies that $\partial_t N(t,u) \neq \partial_u M(t,u)$. ◁

Exact equations (Sect. 2.6).

- Exact differential equations.
- **The Poincaré Lemma.**
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.

The Poincaré Lemma.

Remark: The coefficients N and M of an exact equations are the derivatives of a potential function ψ.

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Lemma (Poincaré)

Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, the continuously differentiable functions $M, N : R \to \mathbb{R}$ satisfy the equation

$$\partial_t N(t,u) = \partial_u M(t,u)$$

iff there exists a twice continuously differentiable function $\psi : R \to \mathbb{R}$, called potential function, *such that for all $(t,u) \in R$ holds*

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($\Rightarrow$) Difficult: Poincaré, 1880.

The Poincaré Lemma.

Example

Show that the function $\psi(t,u) = t^2 + tu^2$ is the potential function for the exact differential equation

$$2ty(t)\,y'(t) + 2t + y^2(t) = 0.$$

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Remark: The Poincaré Lemma only states necessary and sufficient conditions on N and M for the existence of ψ.

Exact equations (Sect. 2.6).

- Exact differential equations.
- The Poincaré Lemma.
- **Implicit solutions and the potential function.**
- Generalization: The integrating factor method.

Implicit solutions and the potential function.

Theorem (Exact differential equations)

Let $M, N : R \to \mathbb{R}$ be continuously differentiable functions on an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$. If the differential equation

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0 \tag{1}$$

is exact, then every solution $y : (t_1, t_2) \to \mathbb{R}$ must satisfy the algebraic equation

$$\psi(t, y(t)) = c,$$

where $c \in \mathbb{R}$ and $\psi : R \to \mathbb{R}$ is a potential function for Eq. (1).

Implicit solutions and the potential function.

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Proof: $0 = N(t, y)\, y' + M(t, y)$

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Implicit solutions and the potential function.

Example

Find all solutions y to the equation

$$\left[\sin(t) + t^2 e^{y(t)} - 1\right] y'(t) + y(t)\cos(t) + 2te^{y(t)} = 0.$$

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These are actually equations for ψ.

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These are actually equations for ψ. From the first one,

$$\psi(t,u) = \int \left[\sin(t) + t^2 e^u - 1\right] du + g(t).$$

Implicit solutions and the potential function.

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Introduce this expression into $\partial_t \psi(t,u) = M(t,u)$,

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Solution: $\psi(t,u) = \int \left[\sin(t) + t^2 e^{u} - 1\right] du + g(t)$. Integrating,

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Introduce this expression into $\partial_t \psi(t,u) = M(t,u)$, that is,

$$\partial_t \psi(t,u) = u\cos(t) + 2te^{u} + g'(t)$$

Implicit solutions and the potential function.

Example

Find all solutions y to the equation

$$\big[\sin(t) + t^2 e^{y(t)} - 1\big]\, y'(t) + y(t)\cos(t) + 2te^{y(t)} = 0.$$

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Therefore, $g'(t) = 0$, so we choose $g(t) = 0$.

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So the solution y satisfies $y(t)\sin(t) + t^2 e^{y(t)} - y(t) = c$. $\lhd$

Exact equations (Sect. 2.6).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- **Generalization: The integrating factor method.**

Remark:
Sometimes a non-exact equation can we transformed into an exact equation multiplying the equation by an integrating factor. Just like in the case of linear differential equations.

Generalization: The integrating factor method.

Theorem (Integrating factor)

Let $M, N : R \to \mathbb{R}$ *be continuously differentiable functions on* $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, *with* $N \neq 0$. *If the equation*

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0$$

is not exact, that is, $\partial_t N(t, u) \neq \partial_u M(t, u)$, *and if the function*

$$\frac{1}{N(t, u)} \big[\partial_u M(t, u) - \partial_t N(t, u)\big]$$

does not depend on the variable u, *then the equation*

$$\mu(t) \big[N(t, y(t))\, y'(t) + M(t, y(t))\big] = 0$$

is exact, where $\dfrac{\mu'(t)}{\mu(t)} = \dfrac{1}{N(t, u)} \big[\partial_u M(t, u) - \partial_t N(t, u)\big]$.

Generalization: The integrating factor method.

Example

Find all solutions y to the differential equation

$$\big[t^2 + t\,y(t)\big]\,y'(t) + \big[3t\,y(t) + y^2(t)\big] = 0.$$

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Find all solutions y to the differential equation

$$\big[t^2 + t\,y(t)\big]\,y'(t) + \big[3t\,y(t) + y^2(t)\big] = 0.$$

Solution: The equation is not exact:

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Find all solutions y to the differential equation

$$\big[t^2 + t\,y(t)\big]\,y'(t) + \big[3t\,y(t) + y^2(t)\big] = 0.$$

Solution: The equation is not exact:

$$N(t,u) = t^2 + tu$$

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$$\frac{\left[\partial_u M(t,u) - \partial_t N(t,u)\right]}{N(t,u)}$$

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$$\frac{\big[\partial_u M(t,u) - \partial_t N(t,u)\big]}{N(t,u)} = \frac{1}{(t^2 + tu)}\big[(3t + 2u) - (2t + u)\big]$$

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We find a function μ solution of $\dfrac{\mu'}{\mu} = \dfrac{\left[\partial_u M - \partial_t N\right]}{N}$,

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$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{t} \quad \Rightarrow \quad \ln(\mu(t)) = \ln(t) \quad \Rightarrow \quad \mu(t) = t.$$

Therefore, the equation below is exact:

$$\left[t^3 + t^2\,y(t)\right] y'(t) + \left[3t^2\,y(t) + t\,y^2(t)\right] = 0.$$

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From the first equation above we obtain

$$\partial_u \psi = t^3 + t^2 u$$

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Introduce ψ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2$. So,

$$\partial_t \psi(t,u) = 3t^2 u + tu^2 + g'(t) = \tilde{M}(t,u)$$

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Introduce ψ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2$. So,

$$\partial_t \psi(t,u) = 3t^2 u + tu^2 + g'(t) = \tilde{M}(t,u) = 3t^2 u + tu^2,$$

Generalization: The integrating factor method.

Example

Find all solutions y to the differential equation

$$\left[t^2 + t\,y(t)\right] y'(t) + \left[3t\,y(t) + y^2(t)\right] = 0.$$

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And every solution y satisfies $t^3\,y(t) + \dfrac{1}{2}\,t^2\big[y(t)\big]^2 = c.$ $\lhd$

Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
- Superposition property.
- Constant coefficients equations.
- The characteristic equation.
- The main result.

Second order linear differential equations.

Definition

Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t) \tag{2}$$

is called a *second order linear* differential equation with *variable coefficients*.

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Remark: The notion of an homogeneous equation presented here is not the same as the notion presented in the previous chapter.

Second order linear differential equations.

Example

(a) A second order, linear, homogeneous, constant coefficients equation is

$$y'' + 5y' + 6 = 0.$$

Second order linear differential equations.

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Second order linear differential equations.

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$$y'' + 2t\, y' - \ln(t)\, y = e^{3t}.$$

(d) Newton's second law of motion $(ma = f)$ for point particles of mass m moving in one space dimension under a force $f : \mathbb{R} \to \mathbb{R}$ is given by

$$m\, y''(t) = f(t).$$

$\triangleleft$

Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
- **Superposition property.**
- Constant coefficients equations.
- The characteristic equation.
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Superposition property.

Theorem

If the functions y_1 *and* y_2 *are solutions to the homogeneous linear equation*

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{3}$$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ *is also a solution for any constants* $c_1, c_2 \in \mathbb{R}$.

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If the functions y_1 and y_2 are solutions to the homogeneous linear equation

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Proof: Verify that the function $y = c_1 y_1 + c_2 y_2$ satisfies Eq. (3) for every constants c_1, c_2,

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$$(c_1 y_1 + c_2 y_2)'' + a_1(t)(c_1 y_1 + c_2 y_2)' + a_0(t)(c_1 y_1 + c_2 y_2)$$

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$$= c_1 \big[y_1'' + a_1(t) y_1' + a_0(t) y_1 \big] + c_2 \big[y_2'' + a_1(t) y_2' + a_0(t) y_2 \big] = 0.$$

□

Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
- Superposition property.
- **Constant coefficients equations.**
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Constant coefficients equations.

Remark: Just by trial and error one can find solutions to second order, constant coefficients, homogeneous, linear differential equations.

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Find solutions to the equation $y'' + 5y' + 6y = 0$.

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Find solutions to the equation $y'' + 5y' + 6y = 0$.

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Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: We look for solutions proportional to exponentials e^{rt}, for an appropriate constant $r \in R$, since the exponential can be canceled out from the equation.

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That is, r must be a root of the polynomial $p(r) = r^2 + 5r + 6$.

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This polynomial is called the characteristic polynomial of the differential equation.

Constant coefficients equations.

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Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

Constant coefficients equations.

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The roots of the characteristic polynomial are

$$r = \frac{1}{2}\left(-5 \pm \sqrt{25 - 24}\right)$$

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$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{-3t}.$$

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Their superposition provides infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

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Summary: The differential equation $y'' + 5y' + 6y = 0$ has infinitely many solutions,

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- There are two free constants in the solution found above.
- The ODE above is second order, so two integrations must be done to find the solution. This explain the origin of the two free constant in the solution.
- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.

Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
- Superposition property.
- Constant coefficients equations.
- **The characteristic equation.**
- The main result.

The characteristic equation.

Definition

Given a second order linear homogeneous differential equation with constant coefficients

$$y'' + a_1 y' + a_0 = 0, \tag{4}$$

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (4) are, respectively,

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

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If r_1, r_2 are the solutions of the characteristic equation and c_1, c_2 are constants, then the function

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is called the *general solution* of the Eq. (4).

The characteristic equation.

Example

Find the solution y of the initial value problem

$$y'' + 5y' + 6 = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

The characteristic equation.

Example

Find the solution y of the initial value problem

$$y'' + 5y' + 6 = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

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We now find the constants c_1 and c_2 that satisfy the initial conditions above:

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Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

◁

The characteristic equation.

Example

Find the general solution y of the differential equation

$$2y'' - 3y' + y = 0.$$

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Solution: We look for every solution of the form $y(t) = e^{rt}$,

The characteristic equation.

Example

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Therefore, the general solution of the equation above is

$$y(t) = c_1 e^{t} + c_2 e^{t/2},$$

where c_1, c_2 are arbitrary constants. $\triangleleft$

Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
- Superposition property.
- Constant coefficients equations.
- The characteristic equation.
- **The main result.**

The main result.

Theorem (Constant coefficients)

Given real constants a_1, a_0, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1\, y' + a_0\, y = 0. \qquad (5)$$

Let r_+, r_- be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let c_0, c_1 be arbitrary constants. Then, any solution of Eq. (5) belongs to only one of the following cases:

(a) *If $r_+ \neq r_-$, the general solution is $y(t) = c_0 e^{r_+ t} + c_1 e^{r_- t}$.*

(b) *If $r_+ = r_- \in \mathbb{R}$, the general solution is $y(t) = (c_0 + c_1 t) e^{r_+ t}$.*

Furthermore, given real constants t_0, y_0 and y_1, there is a unique solution to the initial value problem given by Eq. (5) and the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$