On linear and non-linear equations.(Sect. 2.4).

- Review: Linear differential equations.
- Non-linear differential equations.
- Properties of solutions to non-linear ODE.
- The Bernoulli equation.

Review: Linear differential equations.

Theorem (Variable coefficients)

Given continuous functions $a, b : (t_1, t_2) \to \mathbb{R}$, *with* $t_2 > t_1$, *and given constants* $t_0 \in (t_1, t_2)$, $y_0 \in \mathbb{R}$, *the IVP*

$$y' = -a(t)\,y + b(t), \qquad y(t_0) = y_0,$$

has the unique solution $y : (t_1, t_2) \to \mathbb{R}$ *given by*

$$y(t) = \frac{1}{\mu(t)}\Big[y_0 + \int_{t_0}^{t} \mu(s)\,b(s)\,ds\Big], \tag{1}$$

where the integrating factor function is given by

$$\mu(t) = e^{A(t)}, \qquad A(t) = \int_{t_0}^{t} a(s)\,ds.$$

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Proof: Based on the integration factor method.

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  - (c) For every initial condition $y_0 \in \mathbb{R}$ the corresponding solution $y(t)$ of a linear IVP is defined for all $t \in (t_1, t_2)$.
- None of these properties holds for solutions to non-linear differential equations.

On linear and non-linear equations.(Sect. 2.4).

- Review: Linear differential equations.
- **Non-linear differential equations.**
- Properties of solutions to non-linear ODE.
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Non-linear differential equations.

Definition

An ordinary differential equation $y'(t) = f(t, y(t))$ is called *non-linear* iff the function $(t, u) \mapsto f(t, u)$ is non-linear in the second argument.

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Properties of solutions to non-linear ODE.

Theorem (Non-linear ODE)

Fix a non-empty rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ *and fix a function* $f : R \to \mathbb{R}$ *denoted as* $(t, u) \mapsto f(t, u)$. *If the functions* f *and* $\partial_u f$ *are continuous on* R, *and* $(t_0, y_0) \in R$, *then there exists a smaller open rectangle* $\hat{R} \subset R$ *with* $(t_0, y_0) \in \hat{R}$ *such that the IVP*

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(ii) Non-uniqueness of solution to the IVP above may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.

(iii) Changing the initial data y_0 may change the domain on the variable t where the solution $y(t)$ is defined.

Properties of solutions to non-linear ODE.

Example

Given non-zero constants a_1, a_2, a_3, a_4, find every solution y of

$$y' = \frac{t^2}{\left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right)}.$$

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Integrate, and in the result substitute back the function y:

$$\frac{1}{5}\, y^5(t) + \frac{a_4}{4}\, y^4(t) + \frac{a_3}{3}\, y^3(t) + \frac{a_2}{2}\, y^2(t) + a_1\, y(t) = \frac{t^3}{3} + c.$$

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There is no explicit expression for solutions y of the ODE. $\triangleleft$

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$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

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The first solution is

$$y_1(t) = 0.$$

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$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

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So, the second solution is: $y_2(t) = \Big(\frac{2}{3}t\Big)^{3/2}$.

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$$y(t) = \Big[\frac{2}{3}(t + c)\Big]^{3/2} \Rightarrow 0 = y(0) = \Big(\frac{2}{3}c\Big)^{3/2} \Rightarrow c = 0.$$

So, the second solution is: $y_2(t) = \Big(\frac{2}{3}t\Big)^{3/2}$. Recall $y_1(t) = 0$. $\lhd$

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Find the solution y to the initial value problem

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The solution domain depends on the values of the initial data y_0. $\triangleleft$

Properties of solutions to non-linear ODE.

Summary:

- Linear ODE:
  (a) There is an explicit expression for the solution of a linear IVP.
  (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  (c) The domain of the solution of a linear IVP is defined for every initial condition $y_0 \in \mathbb{R}$.

- Non-linear ODE:
  (i) There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.
  (ii) Non-uniqueness of solution to a non-linear IVP may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.
  (iii) Changing the initial data y_0 may change the domain on the variable t where the solution $y(t)$ is defined.

On linear and non-linear equations.(Sect. 2.4).

- Review: Linear differential equations.
- Non-linear differential equations.
- Properties of solutions to non-linear ODE.
- **The Bernoulli equation.**

The Bernoulli equation.

Remark: The Bernoulli equation is a non-linear differential equation that can be transformed into a linear differential equation.

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Definition

Given functions $p, q : \mathbb{R} \to \mathbb{R}$ and a real number n, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y' + p(t)\, y = q(t)\, y^n$$

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Theorem

The function $y : \mathbb{R} \to \mathbb{R}$ is a solution of the Bernoulli equation for

$$y' + p(t)\, y = q(t)\, y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' - (n-1)p(t)\, v = -(n-1)q(t).$$

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Given arbitrary constants a_0 and b_0, find every solution of the differential equation

$$y' = a_0 y + b_0 y^3.$$

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