The integrating factor method (Sect. 2.1)

- Overview of differential equations.
- Linear Ordinary Differential Equations.
- The integrating factor method.
  - Constant coefficients.
  - The Initial Value Problem.
  - Variable coefficients.

Read:

- The direction field. Example 2 in Section 1.1 in the Textbook.
- See direction field plotters in Internet. For example, see:
  http://math.rice.edu/~dfield/dfpp.html
  This link is given in our class webpage.

Overview of differential equations.

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$$\frac{d^2}{dt^2}\mathbf{x}(t) = \frac{1}{m}\,\mathbf{F}(t, \mathbf{x}(t)),$$

with m the particle mass and $\mathbf{F}$ the force acting on the particle.

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The wave equation is a PDE: The unknown is $u(t, x)$, a function that depends on two variables, and the equation is

$$\frac{\partial^2}{\partial t^2}u(t, x) = v^2\frac{\partial^2}{\partial x^2}u(t, x),$$

with v the wave speed. Sound propagation in air is described by a wave equation, where u represents the air pressure.

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The first order ODE above is called *linear* iff there exist functions $a, b : \mathbb{R} \to \mathbb{R}$ such that $f(t, y) = -a(t)\, y + b(t)$. That is, f is linear on its argument y, hence a first order linear ODE is given by

$$y'(t) = -a(t)\, y(t) + b(t).$$

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Given constants $a, b \in \mathbb{R}$ with $a \neq 0$, the linear differential equation

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has infinitely many solutions, one for each value of $c \in \mathbb{R}$, given by

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$\square$

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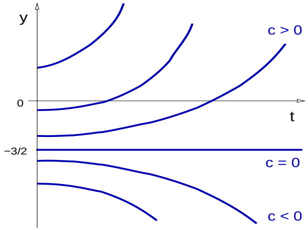

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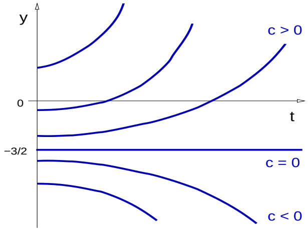

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Verification: $c\, e^{2t} = y + (3/2)$, so $2c\, e^{2t} = y'$, therefore we conclude that y satisfies the ODE $y' = 2y + 3$. $\lhd$

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The *Initial Value Problem* (IVP) for a linear ODE is the following: Given functions $a, b : \mathbb{R} \to \mathbb{R}$ and constants t_0, $y_0 \in R$, find a solution $y : \mathbb{R} \to \mathbb{R}$ of the problem

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Theorem (Constant coefficients)

Given constants $a, b, t_0, y_0 \in \mathbb{R}$, with $a \neq 0$, the initial value problem

$$y' = -ay + b, \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = \Big(y_0 - \frac{b}{a} \Big) e^{-a(t-t_0)} + \frac{b}{a}.$$

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We conclude that $y(t) = \frac{5}{2}\, e^{2t} - \frac{3}{2}$. $\lhd$

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Theorem (Variable coefficients)

Given continuous functions $a, b : \mathbb{R} \to \mathbb{R}$ *and given constants* $t_0, y_0 \in \mathbb{R}$, *the IVP*

$$y' = -a(t)y + b(t) \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = \frac{1}{\mu(t)}\Big[y_0 + \int_{t_0}^{t} \mu(s)b(s)ds\Big],$$

where the integrating factor function is given by

$$\mu(t) = e^{A(t)}, \qquad A(t) = \int_{t_0}^{t} a(s)ds.$$

Remark: See the proof in the Lecture Notes.

The integrating factor method.

Example

Find the solution y to the IVP

$$t\, y' + 2y = 4t^2, \qquad y(1) = 2.$$

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Example

Find the solution y to the IVP

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2.$$

Solution: We first express the ODE as in the Theorem above,

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Find the solution y to the IVP

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$$y' = -\frac{2}{t}\,y + 4t.$$

Therefore, $a(t) = \dfrac{2}{t}$ and $b(t) = 4t$, and also $t_0 = 1$ and $y_0 = 2$.

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Therefore, $a(t) = \dfrac{2}{t}$ and $b(t) = 4t$, and also $t_0 = 1$ and $y_0 = 2$.
We first compute the integrating factor function $\mu = e^{A(t)}$,

The integrating factor method.

Example

Find the solution y to the IVP

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2.$$

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We conclude that $\mu(t) = t^2$.

The integrating factor method.

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Find the solution y to the IVP

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2.$$

Solution: The integrating factor is $\mu(t) = t^2$.

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Find the solution y to the IVP

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2.$$

Solution: The integrating factor is $\mu(t) = t^2$. Hence,

$$t^2\Big(y' + \frac{2}{t}\,y\Big) = t^2(4t)$$

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We conclude that $y(t) = t^2 + \dfrac{1}{t^2}$. ◁

Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- Homogeneous equations.

Separable ODE.

Definition

Given functions $h, g : \mathbb{R} \to \mathbb{R}$, a first order ODE on the unknown function $y : \mathbb{R} \to \mathbb{R}$ is called *separable* iff the ODE has the form

$$h(y)\, y'(t) = g(t).$$

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In textbook: x, $y(x)$ and $M(x) + N(y)\, y'(x) = 0$.

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Therefore: $h(y) = N(y)$

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In textbook: x, $y(x)$ and $M(x) + N(y)\, y'(x) = 0$.

Therefore: $h(y) = N(y)$ and $g(t) = -M(t)$.

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Example

Determine whether the differential equation below is separable,

$$y'(t) = \frac{t^2}{1 - y^2(t)}.$$

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Remark: The functions g and h are not uniquely defined. Another choice here is:

$$g(t) = c\, t^2, \quad h(y) = c\,(1 - y^2), \quad c \in \mathbb{R}.$$

Separable ODE.

Example

Determine whether The differential equation below is separable,

$$y'(t) + y^2(t)\,\cos(2t) = 0$$

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Separable ODE.

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- The differential equation $y'(t) = e^{y(t)} + \cos(t)$ is not separable.

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- The differential equation $y'(t) = e^{y(t)} + \cos(t)$ is not separable.
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Separable ODE.

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- The linear differential equation $y'(t) = -\frac{2}{t}\, y(t) + 4t$ is not separable.
- The linear differential equation $y'(t) = -a(t)\, y(t) + b(t)$, with $b(t)$ non-constant, is not separable.

Separable differential equations (Sect. 2.2).

- Separable ODE.
- **Solutions to separable ODE.**
- Explicit and implicit solutions.
- Homogeneous equations.

Solutions to separable ODE.

Theorem (Separable equations)

If the functions $g, h : \mathbb{R} \to \mathbb{R}$ *are continuous, with* $h \neq 0$ *and with primitives* G *and* H*, respectively; that is,*

$$G'(t) = g(t), \qquad H'(u) = h(u),$$

then, the separable ODE

$$h(y)\, y' = g(t)$$

has infinitely many solutions $y : \mathbb{R} \to \mathbb{R}$ *satisfying the algebraic equation*

$$H(y(t)) = G(t) + c,$$

where $c \in \mathbb{R}$ *is arbitrary.*

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Remark: Given functions g, h, find their primitives G, H.

Solutions to separable ODE.

Example

Find all solutions $y : \mathbb{R} \to \mathbb{R}$ to the ODE $y'(t) = \dfrac{t^2}{1 - y^2(t)}$.

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Find all solutions $y : \mathbb{R} \to \mathbb{R}$ to the ODE $y'(t) = \dfrac{t^2}{1 - y^2(t)}$.

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Then, the Theorem above implies that the solution y satisfies the algebraic equation

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c, \quad c \in \mathbb{R}.$$

$\lhd$

Solutions to separable ODE.

Remarks:

- The equation $y(t) - \dfrac{y^3(t)}{3} = \dfrac{t^3}{3} + c$ is algebraic in y, since there is no y' in the equation.

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- We now verify the previous statement: Differentiate on both sides with respect to t, that is,

$$y'(t) - 3\left(\frac{y^2(t)}{3}\right) y'(t) = 3\,\frac{t^2}{3}$$

Solutions to separable ODE.

Remarks:

- The equation $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is algebraic in y, since there is no y' in the equation.
- Every function y satisfying the algebraic equation

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c,$$

  is a solution of the differential equation above.
- We now verify the previous statement: Differentiate on both sides with respect to t, that is,

$$y'(t) - 3\left(\frac{y^2(t)}{3}\right) y'(t) = 3\,\frac{t^2}{3} \quad \Rightarrow \quad (1 - y^2)\, y' = t^2.$$

Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
- **Explicit and implicit solutions.**
- Homogeneous equations.

Explicit and implicit solutions.

Remark:

The solution $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is given in implicit form.

Explicit and implicit solutions.

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Definition

Assume the notation in the Theorem above. The solution y of a separable ODE is given in *implicit form* iff function y is specified by

$$H\big(y(t)\big) = G(t) + c,$$

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The solution y of a separable ODE is given in *explicit form* iff function H is invertible and y is specified by

$$y(t) = H^{-1}\big(G(t) + c\big).$$

Explicit and implicit solutions.

Example

Use the main idea in the proof of the Theorem above to find the solution of the IVP

$$y'(t) + y^2(t)\cos(2t) = 0, \qquad y(0) = 1.$$

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$$\frac{y'(t)}{y^2(t)} = -\cos(2t)$$

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Substitute the unknown function y back in the equation above,

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$$-\frac{1}{y(t)} = -\frac{1}{2}\sin(2t) + c. \qquad \text{(Implicit form.)}$$

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The initial condition implies that $1 = y(0) = \frac{2}{0 - 2c}$, so $c = -1$.

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We conclude that $y(t) = \dfrac{2}{\sin(2t) + 2}$. ◁

Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- **Homogeneous equations.**

Homogeneous equations.

Definition

The first order ODE $y'(t) = f\big(t, y(t)\big)$ is called *homogeneous* iff for every numbers $c, t, u \in \mathbb{R}$ the function f satisfies

$$f(ct, cu) = f(t, u).$$

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- The function f is invariant under the change of scale of its arguments.

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- So, there exists $F : \mathbb{R} \to \mathbb{R}$ such that $f(t, u) = F\Big(\dfrac{u}{t}\Big)$.

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- If $f(t, u)$ has the property above, it must depend only on u/t.
- So, there exists $F : \mathbb{R} \to \mathbb{R}$ such that $f(t, u) = F\Big(\dfrac{u}{t}\Big)$.
- Therefore, a first order ODE is homogeneous iff it has the form

$$y'(t) = F\Big(\frac{y(t)}{t}\Big).$$

Homogeneous equations.

Example

Show that the equation below is homogeneous,

$$(t - y)\, y' - 2y + 3t + \frac{y^2}{t} = 0.$$

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Divide numerator and denominator by t.

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Divide numerator and denominator by t. We get,

$$y' = \frac{\left(2y - 3t - \dfrac{y^2}{t}\right)}{(t - y)} \, \frac{\left(\dfrac{1}{t}\right)}{\left(\dfrac{1}{t}\right)}$$

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Indeed, in our case:

$$f(t, y) = \frac{2y - 3t - (y^2/t)}{t - y},$$

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Indeed, in our case:

$$f(t,y) = \frac{2y - 3t - (y^2/t)}{t-y}, \qquad F(x) = \frac{2x - 3 - x^2}{1-x},$$

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Indeed, in our case:

$$f(t, y) = \frac{2y - 3t - (y^2/t)}{t - y}, \qquad F(x) = \frac{2x - 3 - x^2}{1 - x},$$

and $f(t, y) = F(y/t)$. $\lhd$

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Determine whether the equation below is homogeneous,

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Divide numerator and denominator by t^3,

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Solution:

Divide numerator and denominator by t^3, we obtain

$$y' = \frac{t^2}{(1 - y^3)} \frac{\left(\frac{1}{t^3}\right)}{\left(\frac{1}{t^3}\right)}$$

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Solution:

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We conclude that the differential equation is not homogeneous. $\triangleleft$

Homogeneous equations.

Theorem

If the differential equation $y'(t) = f\big(t, y(t)\big)$ *is homogeneous, then the differential equation for the unknown* $v(t) = \dfrac{y(t)}{t}$ *is separable.*

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Remark: Homogeneous equations can be transformed into separable equations.

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Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function F.

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Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function F. Introduce $v = y/t$.

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Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function F. Introduce $v = y/t$. This means,

$$y(t) = t\, v(t)$$

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Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function F. Introduce $v = y/t$. This means,

$$y(t) = t\,v(t) \quad \Rightarrow \quad y'(t) = v(t) + t\,v'(t).$$

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Introducing all this into the ODE we get

$$v + t\,v' = F(v)$$

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Remark: Homogeneous equations can be transformed into separable equations.

Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function F. Introduce $v = y/t$. This means,

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Introducing all this into the ODE we get

$$v + t\,v' = F(v) \quad \Rightarrow \quad v' = \frac{\big(F(v) - v\big)}{t}.$$

This last equation is separable. □

Homogeneous equations.

Example

Find all solutions y of the ODE $y' = \dfrac{t^2 + 3y^2}{2ty}$.

Homogeneous equations.

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Find all solutions y of the ODE $y' = \dfrac{t^2 + 3y^2}{2ty}$.

Solution: The equation is homogeneous, since

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Therefore, we introduce the change of unknown $v = y/t$, so $y = t\,v$ and $y' = v + t\,v'$. Hence

$$v + t\,v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t\,v' = \frac{1 + 3v^2}{2v} - v$$

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$$y' = \frac{t^2 + 3y^2}{2ty} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad \Rightarrow \quad y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$

Therefore, we introduce the change of unknown $v = y/t$, so $y = t\,v$ and $y' = v + t\,v'$. Hence

$$v + t\,v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t\,v' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v}$$

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We obtain the separable equation $v' = \dfrac{1}{t}\left(\dfrac{1 + v^2}{2v}\right)$.

Homogeneous equations.

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Find all solutions y of the ODE $y' = \dfrac{t^2 + 3y^2}{2ty}$.

Solution: Recall: $v' = \dfrac{1}{t}\left(\dfrac{1+v^2}{2v}\right)$.

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Find all solutions y of the ODE $y' = \dfrac{t^2 + 3y^2}{2ty}$.

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$$\frac{2v}{1+v^2}\, v' = \frac{1}{t}$$

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But $u = e^{\ln(t)} e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. Hence

$$1 + v^2 = c_1 t \quad \Rightarrow \quad 1 + \left(\frac{y}{t}\right)^2 = c_1 t \quad \Rightarrow \quad y(t) = \pm t\sqrt{c_1 t - 1}.$$

Modeling with first order equations (Sect. 2.3).

- Main example: Salt in a water tank.
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.

Salt in a water tank.

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- To construct a model means to find the differential equation that takes into account the above properties of the system.
- Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.

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- **Main example: Salt in a water tank.**
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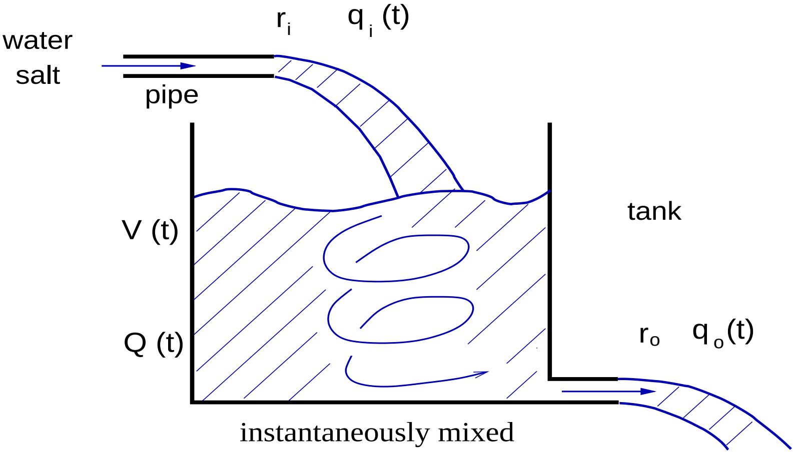
The experimental device.



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Modeling with first order equations (Sect. 2.3).

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The main equations.

Remarks:

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$$\left[r_i q_i - r_o q_o\right] = \frac{\text{Volume}}{\text{Time}} \frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}.$$

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Analysis of the mathematical model.

Eqs. (4) and (1) imply

$$V(t) = (r_i - r_o)\, t + V_0, \tag{5}$$

where $V(0) = V_0$ is the initial volume of water in the tank.

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Analysis of the mathematical model.

Recall: $\dfrac{d}{dt}Q(t) = r_i\, q_i(t) - \dfrac{r_o}{(r_i - r_o)\, t + V_0}\, Q(t).$

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Modeling with first order equations (Sect. 2.3).

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Example

Assume that $r_i = r_o = r$ and q_i are constants.
If r, q_i, Q_0 and V_0 are given, find $Q(t)$.

Predictions for particular situations.

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Solution: Always holds $Q'(t) = -a(t)\, Q(t) + b(t)$.

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.
If r, q_i, Q_0 and V_0 are given, find $Q(t)$.

Solution: Always holds $Q'(t) = -a(t)\,Q(t) + b(t)$.
In this case:

$$a(t) = \frac{r_o}{(r_i - r_o)\,t + V_0}$$

Predictions for particular situations.

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We need to solve the IVP:

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We need to solve the IVP:

$$Q'(t) = -a_0\, Q(t) + b_0, \quad Q(0) = Q_0.$$

Predictions for particular situations.

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Assume that $r_i = r_o = r$ and q_i are constants.
If r, q_i, Q_0 and V_0 are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) = -a_0\,Q(t) + b_0, \quad Q(0) = Q_0$.

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Integrating factor method:

Predictions for particular situations.

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$$A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad Q(t) = \frac{1}{\mu(t)} \Big[Q_0 + \int_0^t \mu(s)\, b_0\, ds \Big].$$

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$$\int_0^t \mu(s)\, b_0\, ds = \frac{b_0}{a_0} \big(e^{a_0 t} - 1 \big)$$

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So: $Q(t) = \Big(Q_0 - \dfrac{b_0}{a_0} \Big) e^{-a_0 t} + \dfrac{b_0}{a_0}$.

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Assume that $r_i = r_o = r$ and q_i are constants.
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So: $Q(t) = \Big(Q_0 - \dfrac{b_0}{a_0} \Big) e^{-a_0 t} + \dfrac{b_0}{a_0}$. But $\dfrac{b_0}{a_0} = r q_i \dfrac{V_0}{r}$

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We conclude: $Q(t) = \big(Q_0 - q_i V_0 \big)\, e^{-rt/V_0} + q_i V_0$.

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Particular cases:

- $\dfrac{Q_0}{V_0} > q_i$;
- $\dfrac{Q_0}{V_0} = q_i$, so $Q(t) = Q_0$;
- $\dfrac{Q_0}{V_0} < q_i$.

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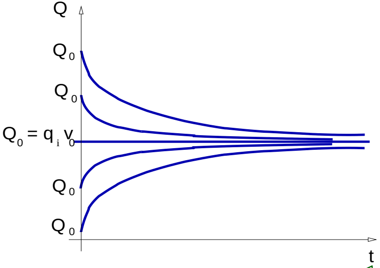

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Predictions for particular situations.

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example.

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Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$,

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$$Q(t) = Q_0\, e^{-rt/V_0}.$$

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Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o)\, t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.

Predictions for particular situations.

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o)\, t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.
So $q(t) = Q(t)/V(t)$ is given by $q(t) = \dfrac{Q_0}{V_0}\, e^{-rt/V_0}$.

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o)\, t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.
So $q(t) = Q(t)/V(t)$ is given by $q(t) = \dfrac{Q_0}{V_0}\, e^{-rt/V_0}$. Therefore,

$$\frac{1}{100}\,\frac{Q_0}{V_0} = q(t_1)$$

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

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So $q(t) = Q(t)/V(t)$ is given by $q(t) = \dfrac{Q_0}{V_0}\, e^{-rt/V_0}$. Therefore,

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Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = \big(Q_0 - q_i V_0\big)\, e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

Since $V(t) = (r_i - r_o)\, t + V_0$ and $r_i = r_o$, we obtain $V(t) = V_0$.
So $q(t) = Q(t)/V(t)$ is given by $q(t) = \frac{Q_0}{V_0}\, e^{-rt/V_0}$. Therefore,

$$\frac{1}{100}\,\frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0}\, e^{-rt_1/V_0} \quad \Rightarrow \quad e^{-rt_1/V_0} = \frac{1}{100}.$$

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$.

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big)$$

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100)$$

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If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0}\,t_1 = \ln(100).$$

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \dfrac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0}\,t_1 = \ln(100).$$

We conclude that $t_1 = \dfrac{V_0}{r}\,\ln(100)$.

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Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \frac{1}{100}$. Then,

$$-\frac{r}{V_0}\,t_1 = \ln\Big(\frac{1}{100}\Big) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0}\,t_1 = \ln(100).$$

We conclude that $t_1 = \frac{V_0}{r}\ln(100)$.

In this case: $t_1 = 100\ln(100)$. $\triangleleft$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ are constants. If $r = 5x10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

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Assume that $r_i = r_o = r$ are constants. If $r = 5x10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

Solution: Recall: $Q'(t) = -a(t)\,Q(t) + b(t)$.

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Assume that $r_i = r_o = r$ are constants. If $r = 5x10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

Solution: Recall: $Q'(t) = -a(t)\,Q(t) + b(t)$. In this case:

$$a(t) = \frac{r_o}{(r_i - r_o)\,t + V_0}$$

Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ are constants. If $r = 5x10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

Solution: Recall: $Q'(t) = -a(t)\,Q(t) + b(t)$. In this case:

$$a(t) = \frac{r_o}{(r_i - r_o)\,t + V_0} \quad \Rightarrow \quad a(t) = \frac{r}{V_0} = a_0,$$

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We conclude: $Q(t) = re^{-rt/V_0} \displaystyle\int_0^t e^{rs/V_0}\,\big[2 + \sin(2s)\big]\,ds.$