Review for Final Exam. Chapters 3, 2.

- Systems of linear Equations (Chptr. 7).
- Laplace transforms (Chptr. 6).
- Power series solutions (Chptr. 5).
- Second order linear equations (Chptr. 3).
- First order differential equations (Chptr. 2).

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- **Second order linear equations (Chptr. 3).**
- First order differential equations (Chptr. 2).

Second order linear equations (Chptr. 3).

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t} \cos(\beta t), \qquad y_2(t) = e^{\alpha t} \sin(\beta t).$$

If $r_1 = r_2 = r$, real, then the general solution is

$$y(t) = (c_1 + c_2 t)\, e^{rt}.$$

Second order linear equations (Chptr. 3).

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1' = -\frac{y_2 g}{W}, \qquad u_2' = \frac{y_1 g}{W}.$$

Second order linear equations (Chptr. 3).

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\, y_1.$$

Choose $c_1 = 0$, $c_2 = 1$. Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. $\triangleleft$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_\pm = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

But this $y_p = k\,e^{-t}$ is solution of the homogeneous equation.

Then propose $y_p(t) = kt\,e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3 \;\Rightarrow\; k = -\frac{3}{4}.$$

We obtain: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

$$\left.\begin{array}{r} c_1 + c_2 = 1, \\ 3_1 - c_2 = 1 \end{array}\right\} \;\Rightarrow\; \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Second order linear equations (Chptr. 3).

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, we obtain,

$$y(t) = \frac{1}{2}\left(e^{3t} + e^{-t}\right) - \frac{3}{4}t\,e^{-t}.$$

$\triangleleft$

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- **First order differential equations (Chptr. 2).**

First order differential equations (Chptr. 2).

Summary:

- Linear, first order equations: $y' + p(t)\, y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\, dt}$.

- Separable, non-linear equations: $h(y)\, y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\, dt$, that is,

  $$\int h(u)\, du = \int g(t)\, dt + c.$$

  The solution can be found in implicit of explicit form.

- Homogeneous equations can be converted into separable equations.

  Read page 49 in the textbook.

- No modeling problems from Sect. 2.3.

First order differential equations (Chptr. 2).

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

  $$N(x,y)\, y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ, such that $N = \partial_y \psi$ and $M = \partial_x \psi$.

  The solution of the differential equation is

  $$\psi\big(x, y(x)\big) = c.$$

First order differential equations (Chptr. 2).

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)\,y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\,y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)
5. Exact equations.
   (Check one equation: $N\,y' + M = 0$, and $\partial_t N = \partial_y M$.)
6. Exact equation with integrating factor.
   (Very complicated to check.)

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy}\,\frac{(1/x^2)}{(1/x^2)} \quad\Rightarrow\quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad\Rightarrow\quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

$$x\,v' = \frac{1 + v + v^2}{v} - v = \frac{1 + v + v^2 - v^2}{v} \quad\Rightarrow\quad x\,v' = \frac{1 + v}{v}.$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c \quad \Rightarrow \quad 1 + v - \ln|1+v| = \ln|x| + c.$$

$$v = \frac{y}{x} \quad \Rightarrow \quad 1 + \frac{y(x)}{x} - \ln\Big|1 + \frac{y(x)}{x}\Big| = \ln|x| + c. \qquad \triangleleft$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method. $\mu(x) = e^{-2x}$.

$$e^{-2x}\, v' - 2\, e^{-2x}\, v = 2 \quad \Rightarrow \quad \big(e^{-2x}\, v\big)' = 2.$$

First order differential equations (Chptr. 2).

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm\frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \quad \Rightarrow \quad c = 9 \quad \Rightarrow \quad y(x) = \frac{e^{-x}}{\sqrt{2x + 9}}. \quad \lhd$$

First order differential equations (Chptr. 2).

Example

Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2 y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2 y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \;\Rightarrow\; \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2 y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2 y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \;\Rightarrow\; g'(x) = 0.$$

$$\psi(x, y) = x^2 y^2 + 2xy + c, \quad x^2\, y^2(x) + 2x\, y(x) + c = 0. \quad \lhd$$