Review for Final Exam. Chapters 7, 6, 5.

- Systems of linear Equations (Chptr. 7).
- Laplace transforms (Chptr. 6).
- Power series solutions (Chptr. 5).
- Second order linear equations (Chptr. 3).
- First order differential equations (Chptr. 2).

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm ie^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

$$\mathbf{x}^{(1)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big],$$

$$\mathbf{x}^{(2)} = e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Systems of linear Equations (Chptr. 7).

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t}, \quad \mathbf{x}^{(2)} = (\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Then, the general solution is

$$\mathbf{x} = c_1\,\mathbf{v}\,e^{\lambda t} + c_2\,(\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Systems of linear Equations (Chptr. 7). FE June 13, 2008.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3,\ \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$
The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

We conclude: $\mathbf{x}(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$ $\triangleleft$

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- Power series solutions (Chptr. 5).
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Laplace transforms (Chptr. 6).

Summary:

- Main Properties:

$$\mathcal{L}\big[f^{(n)}(t)\big] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \quad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \quad (14)$$

- Convolutions:

$$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)].$$

- Partial fraction decompositions, completing the squares.

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2+9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

$$\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

$$\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9)+(bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9}, \quad c = 0, \quad b = -a \quad \Rightarrow \quad b = -\frac{1}{9}.$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(e^{-5s}\,\mathcal{L}[u(t)] - e^{-5s}\,\mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Laplace transforms (Chptr. 6). FE June 13, 2008.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution:

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Therefore, we conclude that,

$$y(t) = 3\cos(3t) + \frac{2}{3}\sin(3t) + \frac{u_5(t)}{9}\Big[1 - \cos(3(t-5))\Big].$$

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Power series solutions (Chptr. 5).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

(c) Euler equation: $(x - x_0)^2\, y'' + \alpha\, (x - x_0)\, y' + \beta\, y = 0$.

Solutions: If $y(x) = |x - x_0|^r$, then r is solution of the indicial equation $p(r) = r(r-1) + \alpha r + \beta = 0$.

Power series solutions (Chptr. 5).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\, (x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\, |x - x_0|^{r_1} + c_2\, |x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_\pm = \lambda \pm \mu i$. Then, the real-valued general solution is

$$y(x) = c_1\, |x - x_0|^{\lambda} \cos\big(\mu \ln |x - x_0|\big) + c_2\, |x - x_0|^{\lambda} \sin\big(\mu \ln |x - x_0|\big).$$

(iii) If $r_1 = r_2 = r$, real, then the general solution is

$$y(x) = \big(c_1 + c_2 \ln |x - x_0|\big)\, |x - x_0|^r.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)} \quad \Rightarrow \quad -3y = \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)}.$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)}.$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)} + \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)} + \sum_{n=0}^{\infty} a_n x^{(n+1)} = 0.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)} + \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)} + \sum_{n=0}^{\infty} a_n x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{(n-2)} + \sum_{n=1}^{\infty} (-3n) a_n x^{(n-1)} + \sum_{n=0}^{\infty} a_n x^{(n+1)} = 0.$$

$$m = n-2 \qquad m = n-1 \qquad m = n+1$$

$$m \to n \qquad m \to n \qquad m \to n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (-3)(n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty}a_{n-1}\,x^n = 0.$$

$$(2)(1)a_2 + (-3)(1)a_1 + \sum_{n=1}^{\infty}\big[(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1}\big]\,x^n = 0$$

We conclude: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1. \quad \triangleleft$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain $a_3 = \frac{3}{2}\,a_1 - \frac{a_0}{6}$.

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$y(x) = a_0 + a_1x + \frac{3}{2}\,a_1x^2 + \Big(\frac{3}{2}a_1 - \frac{a_0}{6}\Big)\,x^3 + \cdots$$

Power series solutions (Chptr. 5). FE June 13, 2008.

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2}\, a_1 x^2 + \Big(\frac{3}{2} a_1 - \frac{a_0}{6}\Big)\, x^3 + \cdots.$

$$y(x) = a_0\Big(1 - \frac{1}{6}\,x^3 + \cdots\Big) + a_1\Big(x + \frac{3}{2}\,x^2 + \frac{3}{2}\,x^3 + \cdots\Big),$$

We conclude that:

$$y_1(x) = 1 - \frac{1}{6}\,x^3 + \cdots\,,$$

$$y_2(x) = x + \frac{3}{2}\,x^2 + \frac{3}{2}\,x^3 + \cdots\,. \qquad \triangleleft$$