Review of Chapter 7.

- Review of Sections 7.5, 7.6, 7.8.
- Const. Coeff., homogeneours linear differential systems:
  - Real, different eigenvalues (7.5).
  - Complex, different eigenvalues (7.6).
  - Repeated eigenvalues (7.8).

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_\pm = \frac{1}{2}\big[-5 \pm \sqrt{25-16}\big] = \frac{1}{2}\big[-5 \pm 3\big]$$

Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for λ_+.

$$(A+I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$2v_1 = \sqrt{2}\,v_2$. Choosing $v_1 = \sqrt{2}$ and $v_2 = 2$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.
Eigenvector for λ_-.

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$v_1 = -\sqrt{2}\,v_2$. Choosing $v_1 = -\sqrt{2}$ and $v_2 = 1$, so, $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$.

General solution: $\mathbf{x} = c_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$. ◁

Exam: November 12, 2008. Problem 4.

Example

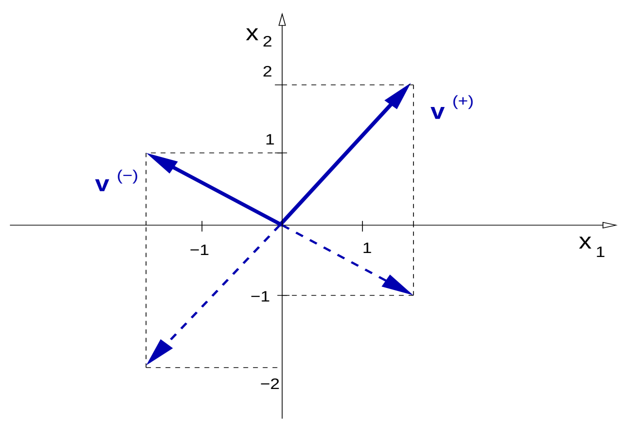
Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

Solution:
We start plotting the vectors

$$\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix},$$

$$\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}.$$



Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

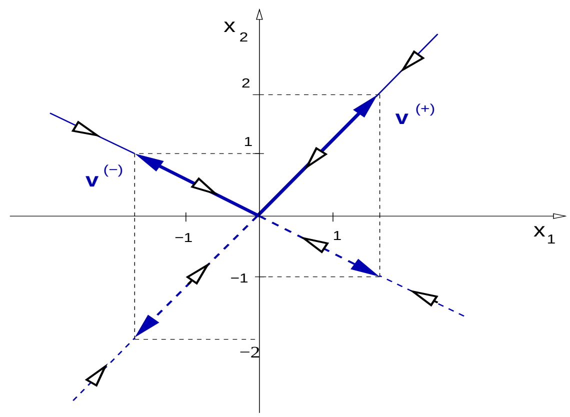
$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

Solution:

We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

$$\mathbf{x}^{(-)}, \quad -\mathbf{x}^{(-)}.$$



Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

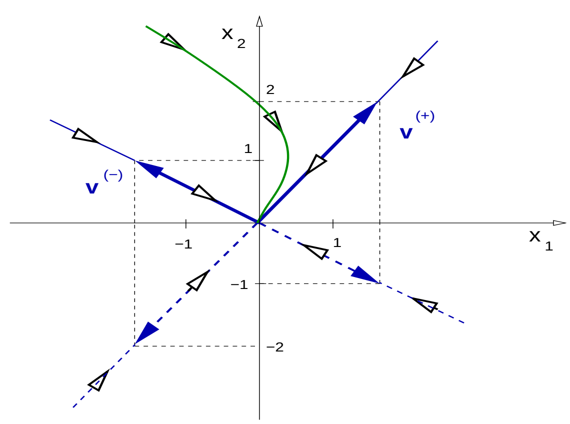
Solution:

Recall: $\lambda_- < \lambda_+ < 0$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)} e^{-t} + \mathbf{v}^{(-)} e^{-4t}.$$



Exam: November 12, 2008. Problem 4.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

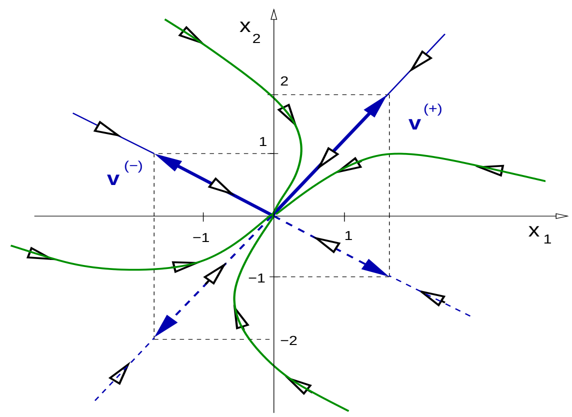
$$\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}, \quad \mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.$$

Solution:

We plot the solutions

$$\mathbf{x} = c_1\,\mathbf{x}^{(+)} + c_2\,\mathbf{x}^{(-)},$$

for $c_1 = \pm 1$ and $c_2 = \pm 1$.



Exam: November 12, 2008. Variation of Problem 4.

Example

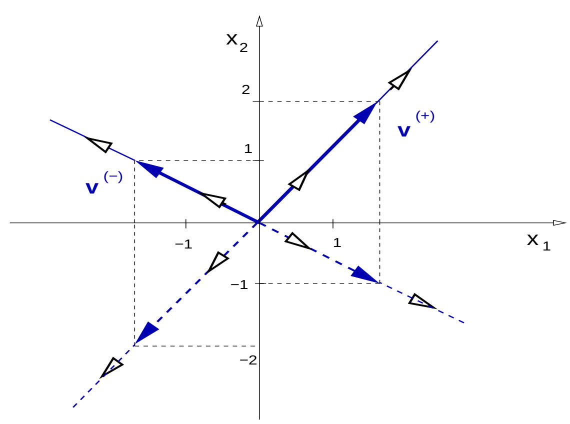
Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.
Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\,e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\,e^{\lambda_- t}$,

Solution:

Here $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

$$\mathbf{x}^{(-)}, \quad -\mathbf{x}^{(-)}.$$



Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = 1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.
Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

Solution:
Recall: $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

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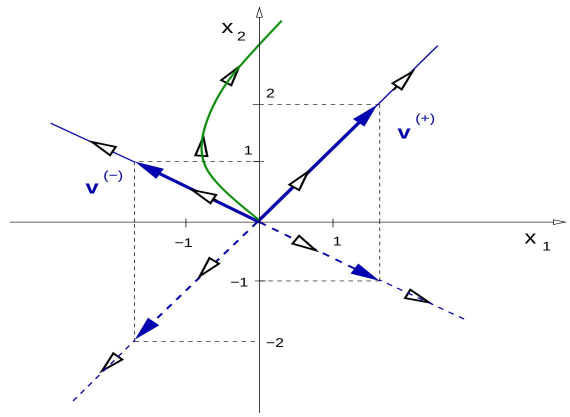

Exam: November 12, 2008. Variation of Problem 4.

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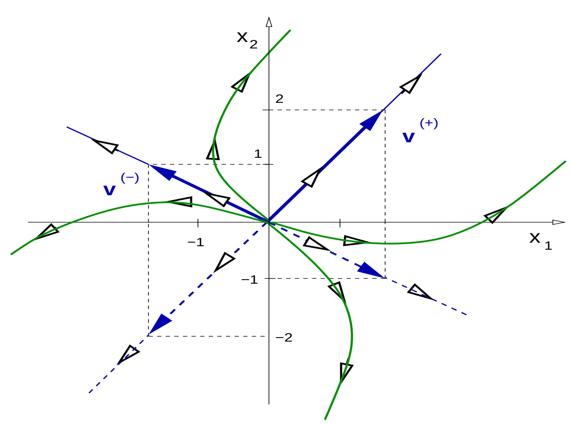
Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

Solution:
Recall: $\lambda_+ > \lambda_- > 0$. We plot the solutions

$$\mathbf{x} = c_1\, \mathbf{x}^{(+)} + c_2\, \mathbf{x}^{(-)},$$

for $c_1 = \pm 1$ and $c_2 = \pm 1$.



Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

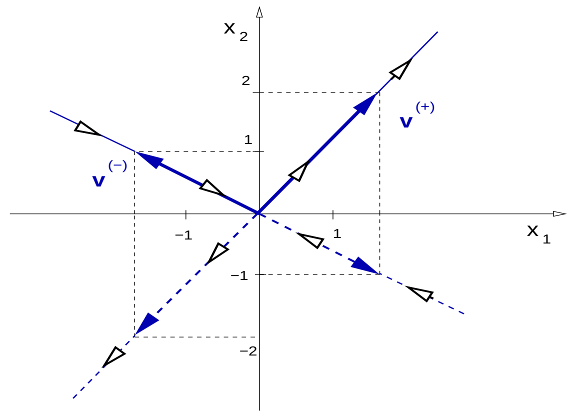
Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

Solution:

Here $\lambda_+ > 0 > \lambda_-$. We plot the solutions

$$\mathbf{x}^{(+)}, \quad -\mathbf{x}^{(+)},$$

$$\mathbf{x}^{(-)}, \quad -\mathbf{x}^{(-)}.$$



Exam: November 12, 2008. Variation of Problem 4.

Example

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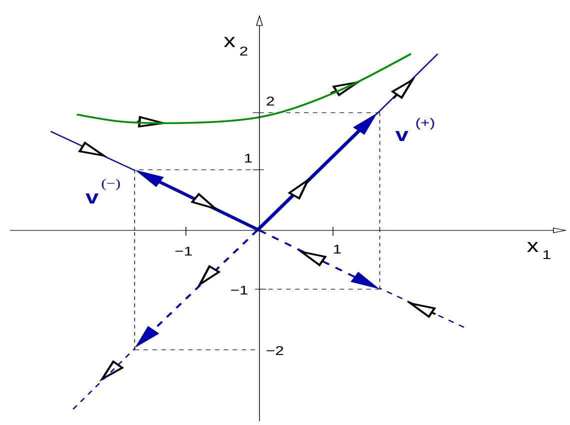
Solution:

Recall: $\lambda_+ > 0 > \lambda_-$. We plot the solutions

$$\mathbf{x} = \mathbf{x}^{(+)} + \mathbf{x}^{(-)},$$

that is,

$$\mathbf{x} = \mathbf{v}^{(+)}\, e^{4t} + \mathbf{v}^{(-)}\, e^{-t}.$$



Exam: November 12, 2008. Variation of Problem 4.

Example

Let $\lambda_+ = 4$, $\lambda_- = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$, and $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

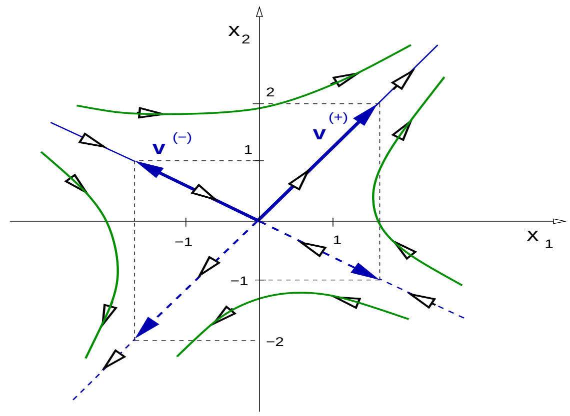
Plot the phase portrait of several linear combinations of the fundamental solutions $\mathbf{x}^{(+)} = v^{(+)}\, e^{\lambda_+ t}$, $\mathbf{x}^{(-)} = v^{(-)}\, e^{\lambda_- t}$,

Solution:

Recall: $\lambda_+ > 0 > \lambda_-$. We plot the solutions

$$\mathbf{x} = c_1\, \mathbf{x}^{(+)} + c_2\, \mathbf{x}^{(-)},$$

for $c_1 = \pm 1$ and $c_2 = \pm 1$.



Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & 4 \\ -1 & (1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4-4}\right] = -1.$$

Hence $\lambda_+ = \lambda_- = -1$. Eigenvector for $\lambda_\pm$.

$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

$v_1 = 2\, v_2$. Choosing $v_1 = 2$ and $v_2 = 1$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Choose $w_2 = 0$, so $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Fundamental sol: $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

General sol: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix}1\\3\end{bmatrix}, \qquad A = \begin{bmatrix}-3 & 4\\-1 & 1\end{bmatrix}.$$

Solution: Recall: $\mathbf{x} = c_1 \begin{bmatrix}2\\1\end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix}2\\1\end{bmatrix} t + \begin{bmatrix}-1\\0\end{bmatrix}\right) e^{-t}.$

Initial condition: $\begin{bmatrix}1\\3\end{bmatrix} = c_1 \begin{bmatrix}2\\1\end{bmatrix} + c_2 \begin{bmatrix}-1\\0\end{bmatrix},$

that is, $\begin{bmatrix}2 & -1\\0 & 1\end{bmatrix} \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$, also, $\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix} \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}.$

The solution is $\mathbf{x} = 3 \begin{bmatrix}2\\1\end{bmatrix} e^{-t} + 5 \left(\begin{bmatrix}2\\1\end{bmatrix} t + \begin{bmatrix}-1\\0\end{bmatrix}\right) e^{-t}.$ $\triangleleft$

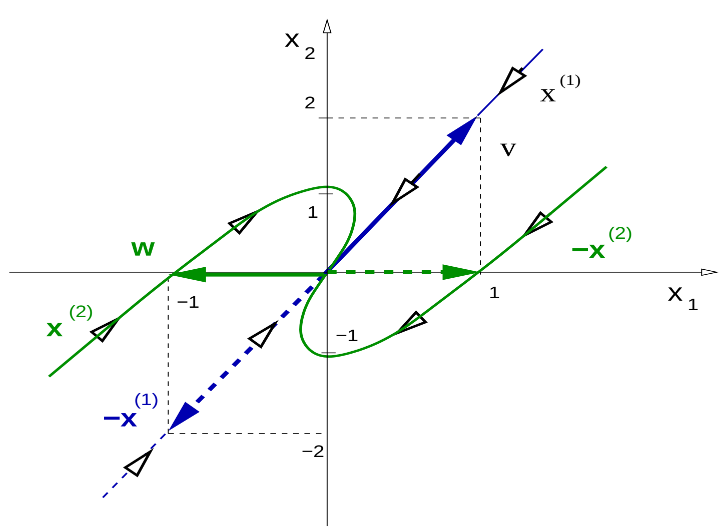
Extra problem.

Example

Let $\lambda = -1$ with $\mathbf{v} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}-1\\0\end{bmatrix}.$

Plot $\pm\mathbf{x}^{(1)} = \pm\mathbf{v}\, e^{-t}$ and $\pm\mathbf{x}^{(2)} = \pm\big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{-t}.$

Solution:



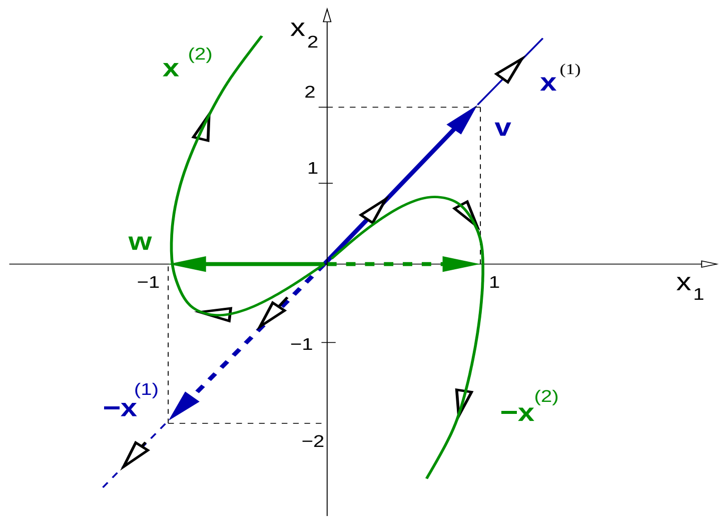
Extra problem.

Example

Let $\lambda = 1$ with $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Plot $\pm\mathbf{x}^{(1)} = \pm\mathbf{v}\, e^{t}$ and $\pm\mathbf{x}^{(2)} = \pm\big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{t}$.

Solution:



Extra problem.

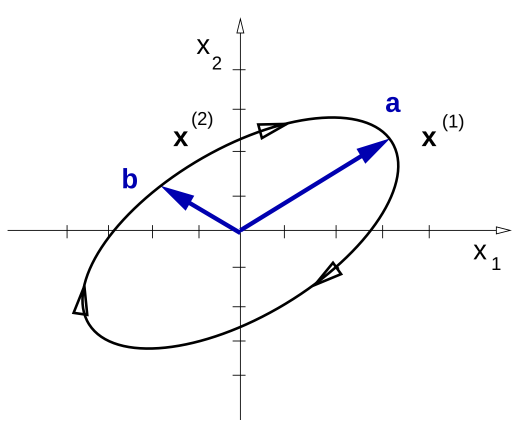
Example

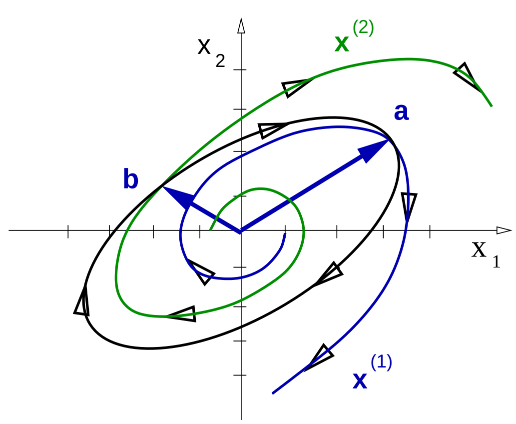
Given any vectors **a** and **b**, sketch qualitative phase portraits of

$\mathbf{x}^{(1)} = \big[\mathbf{a}\,\cos(\beta t) - \mathbf{b}\,\sin(\beta t)\big]\, e^{\alpha t}$, $\mathbf{x}^{(2)} = \big[\mathbf{a}\,\sin(\beta t) + \mathbf{b}\,\cos(\beta t)\big]\, e^{\alpha t}$.

for the cases $\alpha = 0$, and $\alpha > 0$, where $\beta > 0$.

Solution:





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Extra problem.

Example

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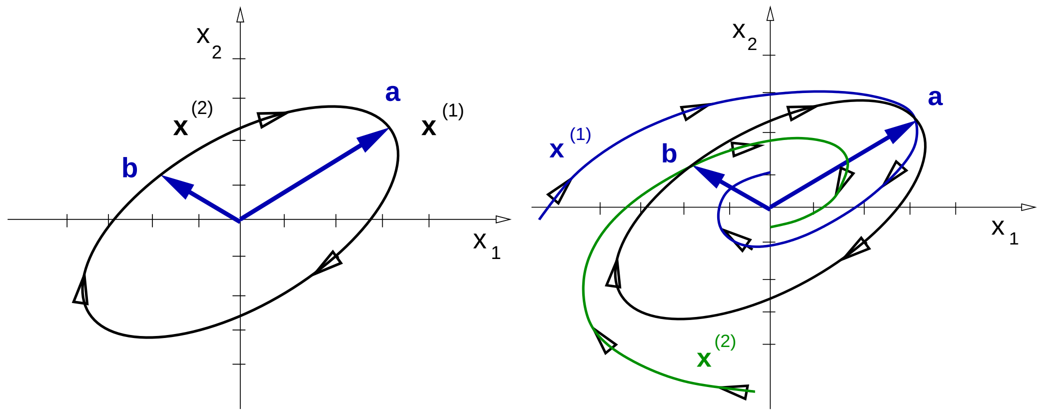
for the cases $\alpha = 0$, and $\alpha < 0$, where $\beta > 0$.

Solution:



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