Review Exam 3.

- Sections 6.1-6.6.
- 5 or 6 problems.
- 50 minutes.
- Laplace Transform table included.

Definition of the Laplace Transform. (Sect 6.1).

Example

Use the definition of the LT to find the LT of $f(t) = \cosh(t)$.

Solution: Recall that $\cosh(t) = (e^t + e^{-t})/2$, and that

$$\mathcal{L}[\cosh(t)] = \int_0^\infty e^{-st}\,\frac{(e^t + e^{-t})}{2}\,dt$$

$$\mathcal{L}[\cosh(t)] = \lim_{N\to\infty} \frac{1}{2}\int_0^N \big(e^{-(s-1)t} + e^{-(s+1)t}\big)\,dt.$$

$$\mathcal{L}[\cosh(t)] = \lim_{N\to\infty} \frac{1}{2}\Big[-\frac{e^{-(s-1)t}}{(s-1)} - \frac{e^{-(s+1)t}}{(s+1)}\Big]\Big|_0^N.$$

$$\mathcal{L}[\cosh(t)] = \frac{1}{2}\Big[\frac{1}{(s-1)} + \frac{1}{(s+1)}\Big] = \frac{1}{2}\,\frac{(s+1)+(s-1)}{(s^2-1)}.$$

We conclude: $\mathcal{L}[\cosh(t)] = \dfrac{s}{s^2-1}$. $\triangleleft$

Computing Laplace Transforms. (Sect 6.2-6.4).

Example

Find the inverse Laplace Transform of $F(s) = \dfrac{e^{-2s}s}{(s-3)^2+25}$.

Solution: We start rewriting function F,

$$F(s) = e^{-2s}\,\frac{s}{(s-3)^2+5^2} = e^{-2s}\,\frac{[(s-3)+3]}{(s-3)^2+5^2}$$

$$F(s) = e^{-2s}\frac{(s-3)}{(s-3)^2+5^2} + e^{-2s}\,\frac{1}{5}\,\frac{5\,(3)}{(s-3)^2+5^2}$$

$$F(s) = e^{-2s}\mathcal{L}[\cos(5t)](s-3) + \frac{3}{5}\,e^{-2s}\,\mathcal{L}[\sin(5t)](s-3)$$

Recall (14): $\mathcal{L}[f(t)](s-c) = \mathcal{L}[e^{ct}\,f(t)]$.

$$F(s) = e^{-2s}\mathcal{L}[e^{3t}\cos(5t)] + \frac{3}{5}\,e^{-2s}\,\mathcal{L}[e^{3t}\,\sin(5t)].$$

Computing Laplace Transforms. (Sect 6.2-6.4).

Example

Find the inverse Laplace Transform of $F(s) = \dfrac{e^{-2s}s}{(s-3)^2+25}$.

Solution:

Recall: $F(s) = e^{-2s}\mathcal{L}[e^{3t}\cos(5t)] + \dfrac{3}{5}\,e^{-2s}\,\mathcal{L}[e^{3t}\,\sin(5t)]$.

Recall (13): $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)\,f(t-c)]$.

$$F(s) = \mathcal{L}[u(t-2)\,e^{3(t-2)}\cos(5(t-2))]$$
$$+\frac{3}{5}\,\mathcal{L}[u(t-2)\,e^{3(t-2)}\sin(5(t-2))].$$

$$f(t) = u(t-2)\,e^{3(t-2)}\left[\cos(5(t-2)) + \frac{3}{5}\,\sin(5(t-2))\right]. \quad \triangleleft$$

Computing Laplace Transforms. (Sect 6.2-6.4).

Example

Find the LT of $f(t) = \begin{cases} \dfrac{t}{2} & 0 \leqslant t \leqslant 6, \\ 3 & t \geqslant 6. \end{cases}$.

Solution: We need to rewrite the function f in terms of functions that apear in the LT table. We need a box function for the first part, and a step function for the second part.

$$f(t) = \frac{t}{2}\big[u(t) - u(t-6)\big] + 3\,u(t-6).$$

$$f(t) = \frac{t}{2}\,u(t) + \Big(-\frac{t}{2} + 3\Big)\,u(t-6) = \frac{t}{2}\,u(t) + \frac{1}{2}(-t+6)\,u(t-6).$$

$$f(t) = \frac{1}{2}\big[t\,u(t) - (t-6)\,u(t-6)\big].$$

Computing Laplace Transforms. (Sect 6.2-6.4).

Example

Find the LT of $f(t) = \begin{cases} \dfrac{t}{2} & 0 \leqslant t \leqslant 6, \\ 3 & t \geqslant 6. \end{cases}$.

Solution: Recall: $f(t) = \dfrac{1}{2}\big[t\,u(t) - (t-6)\,u(t-6)\big]$.

$$\mathcal{L}[f(t)] = \frac{1}{2}\big(\mathcal{L}[t\,u(t)] - \mathcal{L}[(t-6)\,u(t-6)]\big),$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\big(\mathcal{L}[t] - e^{-6s}\mathcal{L}[t]\big),$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\Big(\frac{1}{s^2} - e^{-6s}\frac{1}{s^2}\Big),$$

We conclude that $\mathcal{L}[f(t)] = \dfrac{1}{2s^2}\big(1 - e^{-6s}\big)$. $\triangleleft$

Impulsive forces Sect.(6.5).

Example

(Sect 6.5, $\sim$ Probl.7) Find the solution to the initial value problem

$$y'' + y = \delta(t-\pi)\cos(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Compute the Laplace Transform of the equation,

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)\cos(t)]$$

To compute the right-hand side above, we need the definition of the LT. Given any smooth function f and a constant c, holds

$$\mathcal{L}[\delta(t-c)f(t)] = \int_0^\infty e^{-st}\, f(t)\,\delta(t-c)\,dt = \left[e^{-st} f(t)\right]\Big|_{t=c}$$

We have used that $\displaystyle\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,g(t)\,dt = g(c)$.

We obtain the formula: $\mathcal{L}[\delta(t-c)f(t)] = f(c)\,e^{-cs}$.

Impulsive forces Sect.(6.5).

Example

(Sect 6.5, $\sim$ Probl.7) Find the solution to the initial value problem

$$y'' + y = \delta(t-\pi)\cos(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Recall: $\mathcal{L}[\delta(t-c)f(t)] = f(c)\,e^{-cs}$. Hence

$$s^2\,\mathcal{L}[y] + \mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)\cos(t)] = \cos(\pi)e^{-\pi s} = -e^{-\pi s}$$

$$\mathcal{L}[y] = -e^{-\pi s}\,\frac{1}{s^2+1} = -e^{-\pi s}\,\mathcal{L}\big[\sin(t)\big].$$

Recall the property (13): $e^{-cs}\mathcal{L}[f(t)] = \mathcal{L}\big[u(t-c)\,f(t-c)\big]$;

$$\mathcal{L}[y] = -\mathcal{L}\big[u(t-\pi)\,\sin(t-\pi)\big] \;\Rightarrow\; y(t) = -u(t-\pi)\,\sin(t-\pi).$$

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Convolutions Sect.(6.6).

Example

Given any function $g(t)$ with Laplace transform $G(s) = \mathcal{L}[g(t)]$, find the function f satisfying $\mathcal{L}[f(t)] = \dfrac{e^{-2s}}{(s^2+3)}\, G(s)$.

Solution: One way to solve this is with the splitting

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{(s^2+3)}\,G(s) = e^{-2s}\,\frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s^2+3)}\,G(s),$$

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{\sqrt{3}}\,\mathcal{L}[\sin\big(\sqrt{3}\,t\big)]\,\mathcal{L}[g(t)]$$

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{3}}\mathcal{L}[u_2(t)\,\sin\big(\sqrt{3}\,(t-2)\big)]\,\mathcal{L}[g(t)].$$

$$f(t) = \frac{1}{\sqrt{3}}\int_0^t u_2(\tau)\,\sin\big(\sqrt{3}\,(\tau-2)\big)\,g(t-\tau)1\,d\tau. \qquad \triangleleft$$

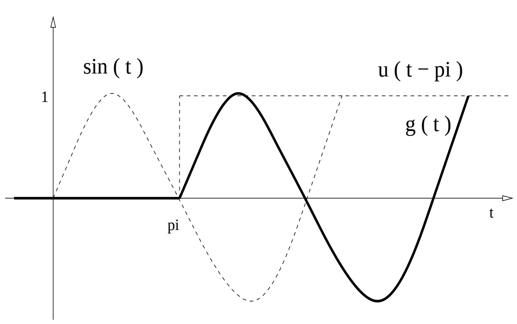
Exam: November 12, 2008. Problem 3

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution:



Express g using step functions,

$$g(t) = u_\pi(t)\,\sin(t-\pi).$$

$$\mathcal{L}[u_c(t)\,f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)].$$

Therefore,

$$\mathcal{L}[g(t)] = e^{-\pi s}\mathcal{L}[\sin(t)].$$

We obtain: $\mathcal{L}[g(t)] = \dfrac{e^{-\pi s}}{s^2+1}$.

Exam: November 12, 2008. Problem 3

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: $\mathcal{L}[g(t)] = \dfrac{e^{-\pi s}}{s^2 + 1}$.

$$\mathcal{L}[y''] - 6\,\mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-\pi s}}{s^2 + 1}.$$

$$(s^2 - 6)\,\mathcal{L}[y] = \frac{e^{-\pi s}}{s^2 + 1} \quad \Rightarrow \quad \mathcal{L}[y] = e^{-\pi s}\,\frac{1}{(s^2 + 1)(s^2 - 6)}.$$

$$H(s) = \frac{1}{(s^2 + 1)(s^2 - 6)} = \frac{1}{(s^2 + 1)(s + \sqrt{6})(s - \sqrt{6})}$$

$$H(s) = \frac{a}{(s + \sqrt{6})} + \frac{b}{(s - \sqrt{6})} + \frac{(cs + d)}{(s^2 + 1)}.$$

Exam: November 12, 2008. Problem 3

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: $H(s) = \dfrac{a}{(s + \sqrt{6})} + \dfrac{b}{(s - \sqrt{6})} + \dfrac{(cs + d)}{(s^2 + 1)}$.

$$\frac{1}{(s^2 + 1)(s + \sqrt{6})(s - \sqrt{6})} = \frac{a}{(s + \sqrt{6})} + \frac{b}{(s - \sqrt{6})} + \frac{(cs + d)}{(s^2 + 1)}$$

$$1 = a(s - \sqrt{6})(s^2 + 1) + b(s + \sqrt{6})(s^2 + 1) + (cs + d)(s^2 - 6).$$

The solution is: $a = -\dfrac{1}{14\sqrt{6}}, \quad b = \dfrac{1}{14\sqrt{6}}, \quad c = 0, \quad d = -\dfrac{1}{7}$.

Exam: November 12, 2008. Problem 3

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: $H(s) = \dfrac{1}{14\sqrt{6}} \Big[-\dfrac{1}{(s + \sqrt{6})} + \dfrac{1}{(s - \sqrt{6})} - \dfrac{2\sqrt{6}}{(s^2 + 1)} \Big].$

$$H(s) = \frac{1}{14\sqrt{6}} \Big[-\mathcal{L}\big[e^{-\sqrt{6}\,t}\big] + \mathcal{L}\big[e^{\sqrt{6}\,t}\big] - 2\sqrt{6}\,\mathcal{L}[\sin(t)] \Big]$$

$$H(s) = \mathcal{L}\Big[\frac{1}{14\sqrt{6}} \Big(-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big) \Big].$$

$$h(t) = \frac{1}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big] \;\Rightarrow\; H(s) = \mathcal{L}[h(t)].$$

Exam: November 12, 2008. Problem 3

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = e^{-\pi s}\, H(s)$, where $H(s) = \mathcal{L}[h(t)]$, and

$$h(t) = \frac{1}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big].$$

$$\mathcal{L}[y] = e^{-\pi s}\,\mathcal{L}[h(t)] = \mathcal{L}[u_\pi(t)\,h(t - \pi)] \;\Rightarrow\; y(t) = u_\pi(t)\,h(t - \pi).$$

Equivalently:

$$y(t) = \frac{u_\pi(t)}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,(t-\pi)} + e^{\sqrt{6}\,(t-\pi)} - 2\sqrt{6}\,\sin(t - \pi) \Big]. \quad \triangleleft$$