Review of Linear Algebra (Sect. 7.3)

- Review: $n \times n$ algebraic linear systems.
- Gauss elimination operations.
- Computing the inverse of an $n \times n$ matrix.
- An application of the determinant.
- Linear dependence of vector sets.

Review: $n \times n$ algebraic linear systems.

Recall: An $n \times n$ algebraic linear system:

Given $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, find $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ solution of

$$A\mathbf{x} = \mathbf{b}.$$

Remark: All the information of the linear system is summarized in the *augmented matrix*,

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right] \quad \Leftrightarrow \quad [A|\mathbf{b}].$$

$n \times n$ systems of linear algebraic equations.

Example

Find the augmented matrix of the system
$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

Solution: The coefficient matrix and the source vector are

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

therefore, the augmented matrix is

$$[A|\mathbf{b}] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right]$$

$\lhd$

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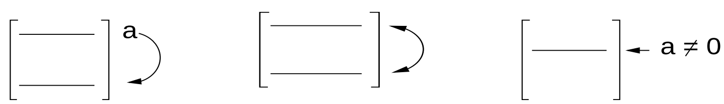
Gauss elimination operations.

Remark: Gauss operations are applied to the augmented matrix of a linear system. They *do* change the augmented matrix but they *do not* change the solutions of the linear system.

(i) Add to one row a multiple of the another;
(ii) Interchange two rows;
(iii) Multiply a row by a non-zero number.

These operations are respectively represented in the picture:



Remark: Use the Gauss operations to find a simple augmented matrix so that the solutions of the linear system are simple to get.

Gauss elimination operations.

Example

Use Gauss operations to find the solution of
$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

Solution: Write down the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 3 & 6 \end{array}\right] \to$$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

The linear system above has the same solution as
$$\begin{aligned} x_1 - 0 &= 1, \\ 0 + x_2 &= 2. \end{aligned}$$

We conclude: $x_1 = 1$ and $x_2 = 2$. $\lhd$

Gauss elimination operations.

Example

Use Gauss operations to find the solution of
$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -\frac{1}{2}x_1 + \frac{1}{4}x_2 &= -\frac{1}{4}. \end{aligned}$$

Solution:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The linear system above has the same solution as
$$\begin{aligned} 2x_1 - x_2 &= 0, \\ 0 &= 1. \end{aligned}$$

We conclude that the system has no solutions. $\lhd$

Remark: If $[A|\mathbf{b}] \to [\tilde{A}|\tilde{\mathbf{b}}]$ having a row $[0, \cdots, 0|1]$, then $A\mathbf{x} = \mathbf{b}$ has no soutions.

Gauss elimination operations.

Example

Find all vectors $\mathbf{b}$ such that the system $A\mathbf{x} = \mathbf{b}$ has solutions, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Solution:

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & -1 & 1 & b_1 + b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 3 & -3 & b_3 - 2b_1 \end{array}\right]$$

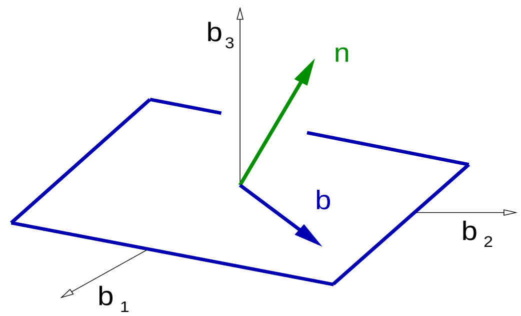
Gauss elimination operations.

Solution:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 3 & -3 & b_3 - 2b_1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_3 + b_1 + 3b_2 \end{array}\right]$$

Therefore: $A\mathbf{x} = \mathbf{b}$ has solutions $\Leftrightarrow$ holds $b_1 + 3b_2 + b_3 = 0$. $\triangleleft$

Remark: All source vectors $\mathbf{b}$ lie on the plane normal to $\mathbf{n} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.



Gauss elimination operations.

Example

Find x_1, x_2, x_3, solution of
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}$$

Solution:

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -3 & 1 & 3 & 24 \\ 0 & 1 & -4 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 7 & 6 & 27 \\ 0 & 1 & -4 & -1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -4 & -1 \\ 0 & 7 & 6 & 27 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 34 & 34 \end{array}\right]$$

Gauss elimination operations.

Example

Find x_1, x_2, x_3, solution of $\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 34 & 34 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}.$$

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Computing the inverse of an $n \times n$ matrix.

Remark: Gauss operations can be used to compute the inverse of a matrix.

The 2×2 case: Find A^{-1} such that $A A^{-1} = I_2$

Denote: $A^{-1} = [\mathbf{x}_1, \ \mathbf{x}_2]$. Then $A [\mathbf{x}_1, \ \mathbf{x}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

That is, $A\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Gauss operations on $\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right]$ and $\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 1 \end{matrix} \end{array}\right]$.

We can solve both systems at once: $\left[\begin{array}{c|c} A & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right]$.

That is, do Gauss operations on $[\,A \mid I_2\,]$.

Computing the inverse of an $n \times n$ matrix.

Example

Use Gauss operations to find the inverse of $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -4 & 1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right]$$

That is, $A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$, or, $A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$. $\triangleleft$

Computing the inverse of an $n \times n$ matrix.

Example

Use Gauss operations to find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

Computing the inverse of an $n \times n$ matrix.

Example

Use Gauss operations to find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

We conclude that $A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. $\triangleleft$

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An application of the determinant.

Theorem

An $n \times n$ matrix A is invertible iff holds $\det(A) \neq 0$.

Example

Is matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$ invertible?

Solution: We only need to compute the determinant of A.

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{vmatrix} = (1) \begin{vmatrix} 5 & 7 \\ 7 & 9 \end{vmatrix} - (2) \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} + (3) \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$$

$$\det(A) = (45 - 49) - 2(18 - 21) + 3(14 - 15) = -4 + 6 - 3$$

Since $\det(A) = -1 \neq 0$, matrix A is invertible. $\triangleleft$

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Linear dependence of vector sets.

Definition

A set of n vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$, with $k \geqslant 1$ is called linearly dependent iff there exists constants $c_1, \cdots, c_k$ with at least one of them non-zero such that

$$c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = \mathbf{0}.$$

Remarks:

(a) Suppose that the nonzero number is c_1. Then

$$\mathbf{v}_1 = -\frac{c_2}{c_1} \mathbf{v}_2 - \cdots - \frac{c_k}{c_1} \mathbf{v}_k.$$

(b) $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is linearly dependent iff one of the vectors is a linear combination of the others.

Linear dependence of vector sets.

Example

(a) A set of two co-linear vectors is linearly dependent.

For example: $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right\}$.

Indeed, $2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$.

(b) A set of three co-planar vectors is linearly dependent.

(c) Any set containing the zero vector is linearly dependent.

For example: $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

Indeed,

$$(1) \begin{bmatrix} 0 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (0) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Linear dependence of vector sets.

Definition

A set of n-vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$, with $k \geqslant 1$ is called linearly independent iff the only linear combination

$$c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = \mathbf{0}.$$

is the one with $c_1 = \cdots = c_k = 0$.

Remark: A non-empty vector set is linearly independent iff the set is not linearly dependent.

Example

Show that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

Solution: Find c_1, c_2 solution of $c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$c_1 - c_2 = 0, \quad 2c_1 + c_2 = 0, \quad 3c_1 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.$$

Linear dependence of vector sets.

Example

Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ linearly independent?

Solution: Find c_1, c_2, c_3 solution of

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} c_3 \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} c_1 + 3c_2 - c_3 &= 0, \\ 2c_1 + 2c_2 + 2c_3 &= 0, \\ 3c_1 + c_2 + 5c_3 &= 0. \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 5 & 0 \end{array}\right].$$

Linear dependence of vector sets.

Example

Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ linearly independent?

Solution:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{aligned} c_1 + 2c_3 &= 0, \\ c_2 - c_3 &= 0. \end{aligned}$$

That is, $c_1 = -2c_3$, $c_2 = c_3$, and c_3 free.

Linear dependence of vector sets.

Example

Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ linearly independent?

Solution: Recall: $c_1 = -2c_3$, $c_2 = c_3$, and c_3 is free.

Since c_3 is free, we choose $c_3 = 1$. Then, we have shown that

$$(-2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We conclude: The vector set is linearly dependent.