Review of Linear Algebra (Sect. 7.2)

- The dot product of n-vectors.
- The matrix-vector product.
- A matrix is a function.
- The inverse of a square matrix.
- The determinant of a square matrix.
- $n \times n$ systems of linear algebraic equations.

The dot product of n-vectors.

Remark: Matrix operations are also true for n-vectors, since n-vectors are $n \times 1$ matrices.

Example

Given $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1+i \\ -1 \\ i \end{bmatrix}$,

compute: (a) $\mathbf{u}^T\mathbf{u}$. (b) $\mathbf{u}^T\mathbf{v}$. (c) $\mathbf{w}^*\mathbf{w}$.

Solution:

(a) $\mathbf{u}^T\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1^2 + 2^2 + 3^2 \quad \Rightarrow \quad \mathbf{u}^T\mathbf{u} = 14.$

The dot product of n-vectors.

Example

Given $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1+i \\ -1 \\ i \end{bmatrix}$,

compute: (a) $\mathbf{u}^T\mathbf{u}$. (b) $\mathbf{u}^T\mathbf{v}$. (c) $\mathbf{w}^*\mathbf{w}$.

Solution:

(b) $\mathbf{u}^T\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -1 + 4 + 3 \quad \Rightarrow \quad \mathbf{u}^T\mathbf{v} = 6.$

(c) $\mathbf{w}^*\mathbf{w} = \begin{bmatrix} 1-i & -1 & -i \end{bmatrix} \begin{bmatrix} 1+i \\ -1 \\ i \end{bmatrix},$

$$\mathbf{w}^*\mathbf{w} = (1^2 + 1^2) + (-1)^2 + 1 \quad \Rightarrow \quad \mathbf{w}^*\mathbf{w} = 4. \qquad \triangleleft$$

The dot product of n-vectors.

Definition

The *dot product* of two real or complex-valued n-vectors $\mathbf{u}$ and $\mathbf{v}$ is the number

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^*\mathbf{v}.$$

Remark: The dot product written in components:

First, express $\mathbf{u}$ and $\mathbf{v}$ in components, $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$

Then compute: $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^*\mathbf{v} = \begin{bmatrix} \overline{u}_1 & \cdots & \overline{u}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$

We conclude: $\mathbf{u} \cdot \mathbf{v} = \overline{u}_1 v_1 + \cdots + \overline{u}_n v_n.$

The dot product of n-vectors.

Remark: The dot product for real-valued vectors:

Then compute: $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.

We conclude: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$.

Remark: Notice that $\mathbf{u} \cdot \mathbf{u} \geqslant 0$, for all $\mathbf{u} \in \mathbb{C}^n$,

since $\mathbf{u} \cdot \mathbf{u} = \mathbf{u}^* \mathbf{u} = \begin{bmatrix} \overline{u}_1 & \cdots & \overline{u}_n \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$,

that is, $\mathbf{u} \cdot \mathbf{u} = |u_1|^2 + \cdots + |u_n|^2 \geqslant 0$.

The dot product of n-vectors.

Remark: The dot product can be used to find the vector length.

Definition

The *norm* or length of an n-vector $\mathbf{u}$ is given by

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}.$$

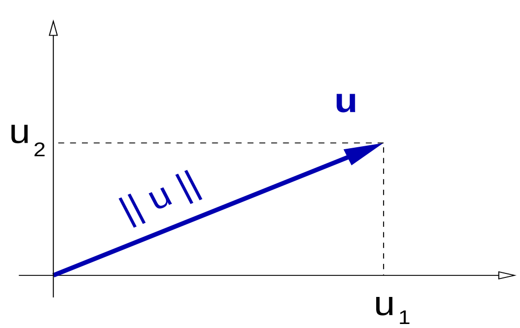
Example

Compute the norm of the real-valued vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

Solution: The norm is,

$$\|\mathbf{u}\| = \sqrt{(u_1)^2 + (u_2)^2}.$$

This is the length of the hypotenuse in the right triangle in the figure. $\lhd$



The dot product of n-vectors.

Remark: Whether two vectors are perpendicular can be determined with the dot product. (We denote $\mathbf{u}$ perpendicular to $\mathbf{v}$ as $\mathbf{u} \perp \mathbf{v}$.)

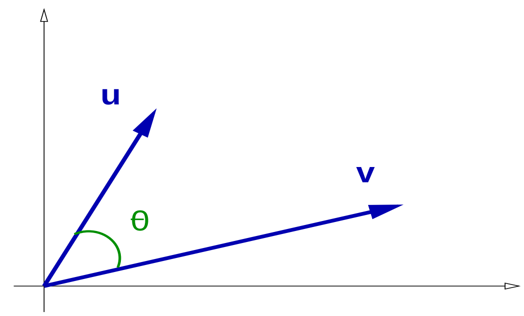
Theorem

$$\mathbf{u} \perp \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} \cdot \mathbf{v} = 0.$$

The proof is based in the formula

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos(\theta),$$

where θ is the smaller angle in between $\mathbf{u}$ and $\mathbf{v}$.



The dot product of n-vectors.

Example

Find a non-zero vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ perpendicular to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

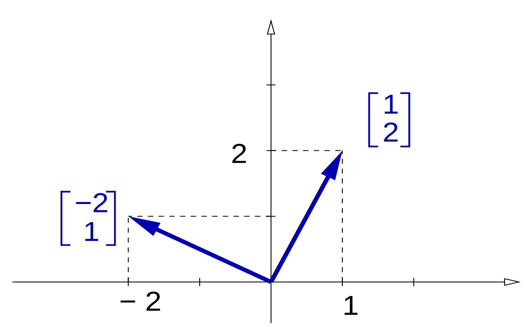
Solution: We need to find u_1 and u_2 solution of

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \quad \Rightarrow \quad u_1 + 2u_2 = 0.$$

So $u_1 = -2u_2$, that is,

$$\mathbf{u} = \begin{bmatrix} -2u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} u_2.$$

We can choose any $u_2 \neq 0$. Say $u_2 = 1$. Therefore, $\mathbf{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. $\triangleleft$



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The matrix-vector product.

Definition

The *matrix-vector product* is the matrix multiplication of an $n \times n$ matrix A and an n-vector $\mathbf{v}$, resulting in an n-vector $A\mathbf{v}$, that is,

$$\begin{array}{cccc} A & \mathbf{v} & \longrightarrow & A\mathbf{v} \\ n \times n & n \times 1 & & n \times 1 \end{array}$$

Example

Find the matrix-vector product $A\mathbf{v}$ for

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Solution: This is a straightforward computation,

$$A\mathbf{v} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-3 \\ -1+6 \end{bmatrix} \quad \Rightarrow \quad A\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}. \quad \lhd$$

The matrix-vector product.

Remark:

- The matrix-vector product provides a new interpretation for a matrix. A matrix is a function.
- An $n \times n$ matrix A is a function $A : \mathbb{R}^n \to \mathbb{R}^n$, given by $\mathbf{v} \mapsto A\mathbf{v}$.
  For example, $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$, is a function that associates $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \to \begin{bmatrix} -1 \\ 5 \end{bmatrix}$, since,
  $$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$
- A matrix is a function, and matrix multiplication is equivalent to function composition.

The matrix-vector product.

Example

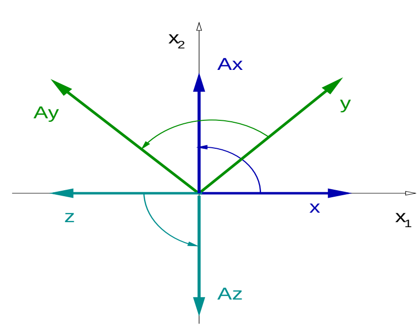
Show that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a rotation in $\mathbb{R}^2$ by $\pi/2$ counterclockwise.

Solution: Matrix A is 2×2, so $A : \mathbb{R}^2 \to \mathbb{R}^2$. Given $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,

$$A\mathbf{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \quad \triangleleft$$



The matrix-vector product.

Definition

An $n \times n$ matrix I_n is called the identity matrix iff holds

$$I_n \mathbf{x} = \mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Example

Write down the identity matrices I_2, I_3, and I_n.

Solution:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}. \quad \triangleleft$$

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The inverse of a square matrix.

Definition

An $n \times n$ matrix A is called *invertible* iff there exists a matrix, denoted as A^{-1}, such

$$(A^{-1})A = I_n, \qquad A(A^{-1}) = I_n.$$

Example

Show that $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ has the inverse $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$.

Solution: We have to compute the product

$$A(A^{-1}) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \Rightarrow \quad A(A^{-1}) = I_2.$$

Check that $(A^{-1})A = I_2$ also holds. $\triangleleft$

The inverse of a square matrix.

Remark: Not every $n \times n$ matrix is invertible.

Theorem (2×2 case)

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff holds that $\Delta = ad - bc \neq 0$. Furthermore, if A is invertible, then

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Verify:

$$A(A^{-1}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \Delta & -ab + ba \\ cd - dc & \Delta \end{bmatrix} = I_2.$$

It is not difficult to see that: $(A^{-1})A = I_2$ also holds.

The inverse of a square matrix.

Example

Find A^{-1} for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution:

We use the formula in the previous Theorem.

In this case: $\Delta = 6 - 2 = 4$, and

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

$\triangleleft$

Remark: The formula for the inverse matrix can be generalized to $n \times n$ matrices having non-zero determinant.

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The determinant of a square matrix.

Definition

The *determinant* of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number

$$\Delta = ad - bc.$$

Notation: The determinant can be denoted in different ways:

$$\Delta = \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Example

(a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$

(b) $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5.$

(c) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0.$ $\triangleleft$

Remark: $\left|\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right|$ is the area of the parallelogram formed by the vectors

$$\begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \begin{bmatrix} b \\ d \end{bmatrix}.$$

The determinant of a square matrix.

Definition

The *determinant* of a 3×3 matrix A is given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Remark: The $|\det(A)|$ is the volume of the parallelepiped formed by the column vectors of A.

The determinant of a square matrix.

Example

Find the determinant of $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

Solution: We use the definition above, that is,

$$\det(A) = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix},$$

$$\det(A) = (1-2) - 3(2-3) - (4-3) = -1 + 3 - 1.$$

We conclude: $\det(A) = 1$. $\triangleleft$

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$n \times n$ systems of linear algebraic equations.

Definition

An *$n \times n$ algebraic system of linear equations* is the following: Given constants a_{ij} and b_i, where indices $i, j = 1 \cdots, n \geqslant 1$, find the constants x_j solutions of the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

The system is called homogeneous iff the sources vanish, that is, $b_1 = \cdots = b_n = 0$.

Example

$$2 \times 2: \quad \begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned} \qquad 3 \times 3: \quad \begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}$$

$\lhd$

$n \times n$ systems of linear algebraic equations.

Remark: Matrix notation is useful to work with systems of linear algebraic equations.

Introduce the coefficient matrix, the source vector, and the unknown vector, respectively,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Using this matrix notation and the matrix-vector product, the linear algebraic system above can be written as

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n. \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}.$$

$n \times n$ systems of linear algebraic equations.

Example

Find the solution to the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Solution: The linear system is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

Since $x_2 = 2x_1$, then $-x_1 + 4x_1 = 3$, that is $x_1 = 1$, hence $x_2 = 2$.

The solution is: $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\triangleleft$