Systems of linear differential equations (Sect. 7.1).

- $n \times n$ systems of linear differential equations.
- Second order equations and first order systems.
- Main concepts from Linear Algebra.

$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton's law of motion for a particle of mass m moving in space. The unknown and the force are vector-valued functions,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \qquad \mathbf{F}(t) = \begin{bmatrix} F_1(t,\mathbf{x}) \\ F_2(t,\mathbf{x}) \\ F_3(t,\mathbf{x}) \end{bmatrix}.$$

The equation of motion are: $m \dfrac{d^2\mathbf{x}}{dt^2} = \mathbf{F}\big(t, \mathbf{x}(t)\big)$.
These are three differential equations,

$$m \frac{d^2 x_1}{dt^2} = F_1\big(t, \mathbf{x}(t)\big), \quad m \frac{d^2 x_2}{dt^2} = F_2\big(t, \mathbf{x}(t)\big), \quad m \frac{d^2 x_3}{dt^2} = F_3\big(t, \mathbf{x}(t)\big).$$

◁

$n \times n$ systems of linear differential equations.

Definition

An *$n \times n$ system of linear first order differential equations* is the following: Given the functions a_{ij}, $g_i : [a, b] \to \mathbb{R}$, where $i, j = 1, \cdots, n$, find n functions $x_j : [a, b] \to \mathbb{R}$ solutions of the n linear differential equations

$$x_1' = a_{11}(t)\, x_1 + \cdots + a_{1n}(t)\, x_n + g_1(t)$$
$$\vdots$$
$$x_n' = a_{n1}(t)\, x_1 + \cdots + a_{nn}(t)\, x_n + g_n(t).$$

The system is called *homogeneous* iff the source functions satisfy that $g_1 = \cdots = g_n = 0$.

$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x_1' = a_{11}(t)\, x_1 + g_1(t).$$

Example

$n = 2$: 2×2 linear system: Find $x_1(t)$ and $x_2(t)$ solutions of

$$x_1' = a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + g_1(t),$$
$$x_2' = a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + g_2(t).$$

Example

$n = 2$: 2×2 homogeneous linear system: Find $x_1(t)$ and $x_2(t)$,

$$x_1' = a_{11}(t)\, x_1 + a_{12}(t)\, x_2$$
$$x_2' = a_{21}(t)\, x_1 + a_{22}(t)\, x_2.$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.$$

Back to x_1 and x_2: $\quad x_1 = \frac{1}{2}(v + w), \qquad x_2 = \frac{1}{2}(v - w).$

We conclude: $\quad x_1(t) = \frac{1}{2}(c_1 + c_2 e^{2t}), \qquad x_2(t) = \frac{1}{2}(c_1 - c_2 e^{2t}).$ $\triangleleft$

Systems of linear differential equations (Sect. 7.1).

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Second order equations and first order systems.

Theorem (Reduction to first order)

Every solution y to the second order linear equation

$$y'' + p(t)\,y' + q(t)\,y = g(t), \tag{1}$$

defines a solution $x_1 = y$ and $x_2 = y'$ of the 2×2 first order linear differential system

$$x_1' = x_2, \tag{2}$$

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t). \tag{3}$$

Conversely, every solution x_1, x_2 of the 2×2 first order linear system in Eqs. (2)-(3) defines a solution $y = x_1$ of the second order differential equation in (1).

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t)\,y - p(t)\,y' + g(t)$. That is,

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t).$$

($\Leftarrow$) Introduce $x_2 = x_1'$ into $x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t)$.

$$x_1'' = -q(t)\,x_1 - p(t)\,x_1' + g(t),$$

that is

$$x_1'' + p(t)\,x_1' + q(t)\,x_1 = g(t).$$

$\square$

Second order equations and first order systems.

Example

Express as a first order system the equation

$$y'' + 2y' + 2y = \sin(at).$$

Solution: Introduce the new unknowns

$$x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2.$$

Then, the differential equation can be written as

$$x_2' + 2x_2 + 2x_1 = \sin(at).$$

We conclude that

$$x_1' = x_2.$$

$$x_2' = -2x_1 - 2x_2 + \sin(at). \qquad \triangleleft$$

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$

$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation: $x_1 = x_2' + x_2$.
Introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,$$

$$x_2'' + x_2' = -x_2' - x_2 + 3x_2,$$

$$x_2'' + 2x_2' - 2x_2 = 0.$$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0.$

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_\pm = -1 \pm \sqrt{3}.$$

Therefore, $x_2 = c_1\, e^{r_+ t} + c_2\, e^{r_- t}$. Since $x_1 = x_2' + x_2$,

$$x_1 = \left(c_1 r_+\, e^{r_+ t} + c_2 r_-\, e^{r_- t}\right) + \left(c_1\, e^{r_+ t} + c_2\, e^{r_- t}\right),$$

We conclude: $x_1 = c_1(1 + r_+)\, e^{r_+ t} + c_2(1 + r_-)\, e^{r_- t}.$ $\triangleleft$

Systems of linear differential equations (Sect. 7.1).

- $n \times n$ systems of linear differential equations.
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- **Main concepts from Linear Algebra.**

Main concepts from Linear Algebra.

Remark: Ideas from Linear Algebra are useful to study systems of linear differential equations.

We review:

- Matrices $m \times n$.
- Matrix operations.
- n-vectors, dot product.
- matrix-vector product.

Definition

An $m \times n$ matrix, A, is an array of numbers

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{array}{c} m \text{ rows,} \\ n \text{ columns.} \end{array}$$

where $a_{ij} \in \mathbb{C}$ and $i = 1, \cdots, m$, and $j = 1, \cdots, n$. An $n \times n$ matrix is called a square matrix.

Main concepts from Linear Algebra.

Example

(a) 2×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) 2×3 matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(c) 3×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(d) 2×2 complex-valued matrix: $A = \begin{bmatrix} 1+i & 2-i \\ 3 & 4i \end{bmatrix}$.

(e) The coefficients of a linear system can be grouped in a matrix,

$$\left.\begin{aligned} x_1' &= -x_1 + 3x_2 \\ x_2' &= x_1 - x_2 \end{aligned}\right\} \quad \Rightarrow \quad A = \begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an m-vector.

Definition

An *m-vector*, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \cdots, m$.

Example

The unknowns of a 2×2 linear system can be grouped in a 2-vector, for example,

$$\left.\begin{aligned} x_1' &= -x_1 + 3x_2 \\ x_2' &= x_1 - x_2 \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}. \qquad \text{Notice that: } \left(A^T\right)^T = A.$$

(b) *A*-conjugate: Conjugate every matrix coefficient:

$$\overline{A} = \begin{bmatrix} 1 & 2-i & -1-2i \\ -3i & 2 & 1 \end{bmatrix}. \qquad \text{Notice that: } \overline{(\overline{A})} = A.$$

Matrix A is real iff $\overline{A} = A$. Matrix A is imaginary iff $\overline{A} = -A$.

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^*)^* = A.$$

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3+5) & (4+1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 5 \end{bmatrix}.$$

The addition $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is not defined.

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \qquad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix} = 4\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Also:

$$\frac{A}{3} = \frac{1}{3}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{2}{3} & \frac{4}{3} & 2 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

$$\begin{array}{ccccc} A & \text{times} & B & \text{defines} & AB \\ m \times n & & n \times \ell & & m \times \ell \end{array}$$

Example: A is 2×2, B is 2×3, so AB is 2×3:

$$AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.$$

Notice B is 2×3, A is 2×2, so BA is not defined.

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{not defined.}$$

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6-2) & (0+1) \\ (-3+4) & (0-2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (6+0) & (-3+0) \\ (4+1) & (-2-2) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}.$$

So $AB \neq BA$. $\triangleleft$

Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find AB for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1-1) & (-1+1) \\ (-1+1) & (1-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\lhd$

Recall: If $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$.

We have just shown that this statement is not true for matrices.