Review for Exam 2.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Regular-singular points (5.5).
  - Euler differential equation (5.4).
  - Power series solutions (5.2).
  - Variation of parameters (3.6).
  - Undetermined coefficients (3.5)
  - Constant coefficients, homogeneous, (3.1)-(3.4).

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Regular-singular points (5.5).

Summary:

- Look for solutions $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)}$.
- Recall: Since $r \neq 0$, holds
  $$y' = \sum_{n=0}^{\infty} (n+r) a_n (x - x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty} (n+r) a_n (x - x_0)^{(n+r-1)},$$
- Find the indicial equation for r, the recurrence relation for a_n.
- Introduce the larger root r_+ of the indicial polynomial into the recurrence relation and solve for a_n.
  - (a) If $(r_+ - r_-)$ is not an integer, then each r_+ and r_- define linearly independent solutions.
  - (b) If $(r_+ - r_-)$ is an integer, then both r_+ and r_- define proportional solutions.

Regular-singular points (5.5).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $y = \sum_{n=0}^{\infty} a_n x^{(n+r)}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r-2)}$,

$$x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)}$$

We also need to compute

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

Regular-singular points (5.5).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $\left(x^2 + \dfrac{1}{4}\right) y = \displaystyle\sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)}$.

Re-label $m = n + 2$ in the first term and then switch back to n,

$$\left(x^2 + \frac{1}{4}\right) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},$$

The equation is

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$

Regular-singular points (5.5).

Example

Consider the equation $x^2\, y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution:

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.$$

$$\left[r(r-1) + \frac{1}{4}\right] a_0\, x^r + \left[(r+1)r + \frac{1}{4}\right] a_1\, x^{(r+1)} +$$

$$\sum_{n=2}^{\infty} \left[(n+r)(n+r-1) a_n + a_{(n-2)} + \frac{1}{4} a_n\right] x^{(n+r)} = 0.$$

Regular-singular points (5.5).

Example

Consider the equation $x^2\,y'' + \left(x^2 + \frac{1}{4}\right)y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $\left[r(r-1) + \dfrac{1}{4}\right] a_0 = 0, \quad \left[(r+1)r + \dfrac{1}{4}\right] a_1 = 0,$

$$\left[(n+r)(n+r-1) + \frac{1}{4}\right] a_n + a_{(n-2)} = 0.$$

The indicial equation $r^2 - r + \dfrac{1}{4} = 0$ implies $r_\pm = \dfrac{1}{2}$.

The indicial equation $r^2 + r + \dfrac{1}{4} = 0$ implies $r_\pm = -\dfrac{1}{2}$.

Choose $r = \dfrac{1}{2}$. That implies a_0 arbitrary and $a_1 = 0$.

Regular-singular points (5.5).

Example

Consider the equation $x^2\,y'' + \left(x^2 + \frac{1}{4}\right)y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \dfrac{1}{2}, \quad a_1 = 0, \quad \left[(n+r)(n+r-1) + \dfrac{1}{4}\right] a_n = -a_{(n-2)}.$

$$\left[\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) + \frac{1}{4}\right] a_n = -a_{(n-2)} \;\Rightarrow\; \left[n^2 - \frac{1}{4} + \frac{1}{4}\right] a_n = -a_{(n-2)}$$

$$n^2 a_n = -a_{(n-2)} \;\Rightarrow\; a_n = -\frac{a_{(n-2)}}{n^2} \;\Rightarrow\; \begin{cases} a_2 = -\dfrac{a_0}{4}, \\[2mm] a_4 = -\dfrac{a_2}{16} = \dfrac{a_0}{64}. \end{cases}$$

Regular-singular points (5.5).

Example

Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: $r = \dfrac{1}{2}$, $a_1 = 0$, $a_2 = -\dfrac{a_0}{4}$, and $a_4 = \dfrac{a_0}{64}$. Then,

$$y(x) = x^r \big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \big).$$

Recall: $a_1 = 0$ and the recurrence relation imply $a_n = 0$ for n odd. Therefore,

$$y(x) = a_0 x^{1/2} \big(1 - \frac{1}{4}\, x^2 + \frac{1}{64}\, x^4 + \cdots \big). \qquad \triangleleft$$

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Euler differential equation (5.4).

Summary:

- $(x - x_0)^2\, y'' + (x - x_0) p_0\, y' + q_0\, y = 0.$
- Find $r_\pm$ solutions of $r(r-1) + p_0 r + q_0 = 0$.
- If $r_+ \neq r_-$ and both are real, then fundamental solutions are

$$y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_-}.$$

- If $r_\pm = \alpha \pm i\beta$, then real-valued fundamental solutions are

$$y_+ = |x - x_0|^\alpha \cos\big(\beta \ln|x - x_0|\big),\ y_- = |x - x_0|^\alpha \sin\big(\beta \ln|x - x_0|\big).$$

- If $r_+ = r_-$ and both are real, then fundamental solutions are

$$y_+ = |x - x_0|^{r_+}, \quad y_- = |x - x_0|^{r_+} \ln|x - x_0|.$$

Euler differential equation (5.4).

Example

Find real-valued fundamental solutions of

$$(x-2)^2\, y'' + 5(x-2)\, y' + 8\, y = 0.$$

Solution: This is an Euler equation. Find r solution of $r(r-1) + 5r + 8 = 0$, that is, $r^2 + 4r + 8 = 0$,

$$r_\pm = \frac{1}{2}\left[-4 \pm \sqrt{16 - 32}\right] \quad \Rightarrow \quad r_\pm = -2 \pm 2i.$$

Real valued fundamental solutions are

$$y_+(x) = |x-2|^{-2} \cos\big(2 \ln|x-2|\big),$$

$$y_-(x) = |x-2|^{-2} \sin\big(2 \ln|x-2|\big). \qquad \triangleleft$$

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Power series solutions (5.2).

Example

Using a power series centered at $x_0 = 0$ find the three first terms of the general solution of $(4 - x^2)\, y'' + 2y = 0$.

Solution: We look for solutions $y = \displaystyle\sum_{n=0}^{\infty} a_n\, x^n$. Therefore,

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)}$$

The differential equation is then given by

$$(4 - x^2) \sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + 2 \sum_{n=0}^{\infty} a_n\, x^n = 0,$$

$$\sum_{n=0}^{\infty} 4n(n-1)a_n\, x^{(n-2)} - \sum_{n=0}^{\infty} n(n-1)a_n\, x^n + \sum_{n=0}^{\infty} 2a_n\, x^n = 0.$$

Power series solutions (5.2).

Example

Using a power series centered at $x_0 = 0$ find the three first terms of the general solution of $(4 - x^2)\,y'' + 2y = 0$.

Solution:

$$\sum_{n=2}^{\infty} 4n(n-1)a_n\,x^{(n-2)} - \sum_{n=0}^{\infty} n(n-1)a_n\,x^n + \sum_{n=0}^{\infty} 2a_n\,x^n = 0.$$

Re-label the first sum, $m = n - 2$ and then switch back to n

$$\sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2}\,x^n - \sum_{n=0}^{\infty} n(n-1)a_n\,x^n + \sum_{n=0}^{\infty} 2a_n\,x^n = 0.$$

$$\sum_{n=0}^{\infty} \big[4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n\big]\,x^n = 0.$$

$$4(n+2)(n+1)a_{n+2} + (-n^2 + n + 2)a_n = 0.$$

Power series solutions (5.2).

Example

Using a power series centered at $x_0 = 0$ find the three first terms of the general solution of $(4 - x^2)\,y'' + 2y = 0$.

Solution: $4(n+2)(n+1)a_{n+2} + (-n^2 + n + 2)a_n = 0.$

Notice: $-n^2 + n + 2 = -(n-2)(n+1)$, hence

$$4(n+2)(n+1)a_{n+2} - (n-2)(n+1)a_n = 0 \quad \Rightarrow \quad a_{n+2} = \frac{(n-2)a_n}{4(n+2)}.$$

For n even the power series terminates at $n = 2$, since

$$a_2 = \frac{-2a_0}{8}, \quad a_4 = 0, \quad a_6 = 0, \cdots$$

For n odd: $a_3 = \dfrac{-a_1}{12}, \quad a_5 = \dfrac{a_3}{20} = -\dfrac{a_1}{(12)(20)}, \cdots$

$$y = a_0\Big[1 - \frac{1}{4}x^2\Big] + a_1\Big[x - \frac{1}{12}\,x^3 - \frac{1}{(12)(20)}\,x^5 + \cdots\Big]. \quad \triangleleft$$

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Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

Solution: We find the solutions of the homogeneous equation,

$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-4 \pm \sqrt{16-16}\right] \quad \Rightarrow \quad r_\pm = -2.$$

Fundamental solutions of the homogeneous equations are

$$y_1 = e^{-2x}, \quad y_2 = x\, e^{-2x}.$$

We now compute their Wronskian,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & x\, e^{-2x} \\ -2e^{-2x} & (1-2x)\, e^{-2x} \end{vmatrix} = (1-2x)\, e^{-4x} + 2x\, e^{-4x}.$$

Hence $W = e^{-4x}$.

Variation of parameters (3.6).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

Solution: $y_1 = e^{-2x}$, $y_2 = x\, e^{-2x}$, $g = x^{-2}\, e^{-2x}$, $W = e^{-4x}$.

Now we find the functions u_1 and u_2,

$$u_1' = -\frac{y_2 g}{W} = -\frac{x\, e^{-2x}\, x^{-2}\, e{-}2x}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|.$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2x}\, x^{-2}\, e{-}2x}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = -\frac{1}{x}.$$

$$y_p = u_1 y_1 + u_2 y_2 = -\ln|x|\, e^{-2x} - \frac{1}{x}\, xe^{-2x} = -(1 + \ln|x|)\, e^{-2x}.$$

Since $\tilde{y}_p = -\ln|x|\, e^{-2x}$ is solution, $y = (c_1 + c_2 x - \ln|x|)\, e^{-2x}$. $\triangleleft$

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Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}$$

Solution: Find the solutions of the homogeneous problem,

$$r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i.$$

$$y_1 = \cos(2x), \quad y_2 = \sin(2x).$$

Start with the first source, $f_1(x) = 3\sin(2x)$.
The function $\tilde{y}_{p_1} = k_1 \sin(2x) + k_2 \cos(2x)$ is the wrong guess, since it is solution of the homogeneous equation. We guess:

$$y_p = x\big[k_1 \sin(2x) + k_2 \cos(2x)\big].$$

$$y_p' = \big[k_1 \sin(2x) + k_2 \cos(2x)\big] + 2x\big[k_1 \cos(2x) - k_2 \sin(2x)\big].$$

$$y_p'' = 4\big[k_1 \cos(2x) - k_2 \sin(2x)\big] + 4x\big[-k_1 \sin(2x) - k_2 \cos(2x)\big].$$

Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}.$$

Solution: Recall: $y_1 = \sin(2x)$, and $y_2 = \cos(2x)$.

$$4\big[k_1 \cos(2x) - k_2 \sin(2x)\big] + 4x\big[-k_1 \sin(2x) - k_2 \cos(2x)\big] +$$
$$4x\big[k_1 \sin(2x) + k_2 \cos(2x)\big] = 3\sin(2x),$$

Therefore, $4\big[k_1 \cos(2x) - k_2 \sin(2x)\big] = 3\sin(2x)$.

Evaluating at $x = 0$ and $x = \pi/4$ we get

$$4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.$$

Therefore, $y_{p_1} = -\frac{3}{4}\, x \cos(2x)$.

Undetermined coefficients (3.5)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}.$$

Solution: Recall: $y_{p_1} = -\frac{3}{4}\, x\cos(2x)$.

We now compute y_{p_2} for $f_2(x) = e^{3x}$.

We guess: $y_{p_2} = k\, e^{3x}$. Then, $y''_{p_2} = 9\, e^{3x}$,

$$(9+4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13} \quad \Rightarrow \quad y_{p_2} = \frac{1}{13}\, e^{3x}.$$

Therefore, the general solution is

$$y(x) = c_1 \sin(2x) + \left(c_2 - \frac{3}{4}\, x\right) \cos(2x) + \frac{1}{13}\, e^{3x}. \qquad \lhd$$