

The Laplace Transform and the IVP (Sect. 6.2).

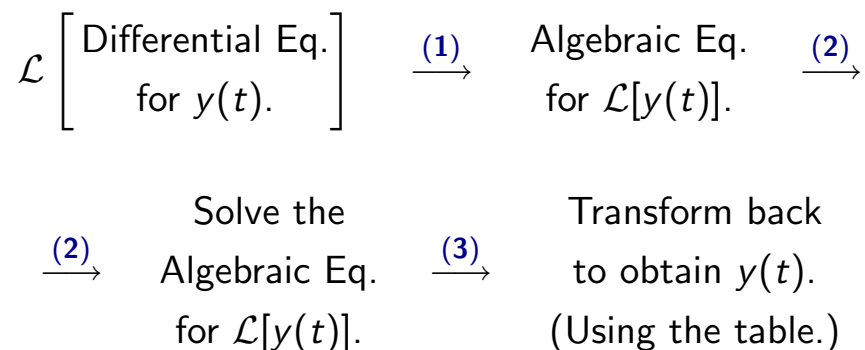
- ▶ Solving differential equations using $\mathcal{L}[\]$.
 - ▶ Homogeneous IVP.
 - ▶ First, second, higher order equations.
 - ▶ Non-homogeneous IVP.

Solving differential equations using $\mathcal{L}[\]$.

Remark: The method works with:

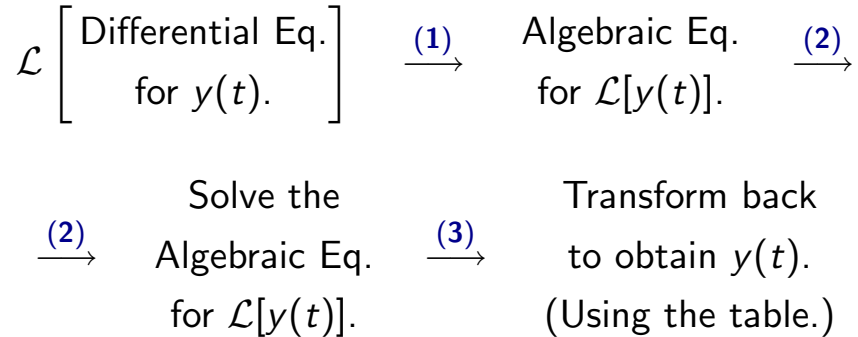
- ▶ Constant coefficient equations.
- ▶ Homogeneous and non-homogeneous equations.
- ▶ First, second, higher order equations.

Idea of the method:



Solving differential equations using $\mathcal{L}[\]$.

Idea of the method:



Recall:

$$(a) \mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)];$$

$$(b) \mathcal{L}[y^{(n)}] = s^n \mathcal{L}[y] - s^{(n-1)} y(0) - s^{(n-2)} y'(0) - \dots - y^{(n-1)}(0).$$

The Laplace Transform and the IVP (Sect. 6.2).

- ▶ Solving differential equations using $\mathcal{L}[\]$.
 - ▶ **Homogeneous IVP.**
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 - ▶ Non-homogeneous IVP.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Compute the $\mathcal{L}[\]$ of the differential equation,

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] \Rightarrow \mathcal{L}[y'' - y' - 2y] = 0.$$

The $\mathcal{L}[\]$ is a linear function, so

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0.$$

Derivatives are transformed into power functions,

$$\left[s^2 \mathcal{L}[y] - s y(0) - y'(0) \right] - \left[s \mathcal{L}[y] - y(0) \right] - 2 \mathcal{L}[y] = 0,$$

We then obtain $(s^2 - s - 2) \mathcal{L}[y] = (s - 1) y(0) + y'(0)$.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Recall: $(s^2 - s - 2) \mathcal{L}[y] = (s - 1) y(0) + y'(0)$.

Differential equation for $y \xrightarrow{\mathcal{L}[\]}$ Algebraic equation for $\mathcal{L}[y]$.

Introduce the initial condition,

$$(s^2 - s - 2) \mathcal{L}[y] = (s - 1).$$

We can solve for the unknown $\mathcal{L}[y]$ as follows,

$$\mathcal{L}[y] = \frac{(s - 1)}{(s^2 - s - 2)}.$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \frac{(s-1)}{(s^2 - s - 2)}$.

The partial fraction method: Find the zeros of the denominator,

$$s^2 - s - 2 = 0 \Rightarrow s_{\pm} = \frac{1}{2}[1 \pm \sqrt{1+8}] \Rightarrow \begin{cases} s_+ = 2, \\ s_- = -1, \end{cases}$$

Therefore, we rewrite: $\mathcal{L}[y] = \frac{(s-1)}{(s-2)(s+1)}$.

Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Recall: $\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}$.

A simple calculation shows

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$$

$$(s-1) = s(a+b) + (a-2b) \Rightarrow \begin{cases} a+b = 1, \\ a-2b = -1 \end{cases}$$

Hence, $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Then, $\mathcal{L}[y] = \frac{1}{3} \frac{1}{(s-2)} + \frac{2}{3} \frac{1}{(s+1)}$.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \frac{1}{3} \frac{1}{(s-2)} + \frac{2}{3} \frac{1}{(s+1)}$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \Rightarrow \frac{1}{s-2} = \mathcal{L}[e^{2t}], \quad \frac{1}{s+1} = \mathcal{L}[e^{-t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \frac{1}{3} \mathcal{L}[e^{2t}] + \frac{2}{3} \mathcal{L}[e^{-t}] = \mathcal{L}\left[\frac{1}{3}(e^{2t} + 2e^{-t})\right]$$

We conclude that: $y(t) = \frac{1}{3}(e^{2t} + 2e^{-t})$. ◁

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Compute the $\mathcal{L}[\]$ of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$

The $\mathcal{L}[\]$ is a linear function,

$$\mathcal{L}[y''] - 4\mathcal{L}[y'] + 4\mathcal{L}[y] = 0.$$

Derivatives are transformed into power functions,

$$\left[s^2 \mathcal{L}[y] - s y(0) - y'(0)\right] - 4 \left[s \mathcal{L}[y] - y(0)\right] + 4 \mathcal{L}[y] = 0,$$

Therefore, $(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4)y(0) + y'(0)$.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4)y(0) + y'(0)$.

Introduce the initial conditions, $(s^2 - 4s + 4) \mathcal{L}[y] = s - 3$.

Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)}$.

The partial fraction method: Find the roots of the denominator,

$$s^2 - 4s + 4 = 0 \Rightarrow s_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 16}] \Rightarrow s_+ = s_- = 2.$$

We obtain: $\mathcal{L}[y] = \frac{(s - 3)}{(s - 2)^2}$.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \frac{(s - 3)}{(s - 2)^2}$. We find the partial fraction,

$$\frac{(s - 3)}{(s - 2)^2} = \frac{a}{s - 2} + \frac{b}{(s - 2)^2} \Rightarrow s - 3 = a(s - 2) + b$$

If $s = 2$, then $b = -1$. If $s = 3$, then $a = 1$. Hence

$$\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2}.$$

From the Laplace transforms table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \Rightarrow \frac{1}{s - 2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s - a)^{(n+1)}} \Rightarrow \frac{1}{(s - 2)^2} = \mathcal{L}[te^{2t}].$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}$ and

$$\frac{1}{s-2} = \mathcal{L}[e^{2t}], \quad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}] = \mathcal{L}[e^{2t} - te^{2t}].$$

We conclude that $y(t) = e^{2t} - te^{2t}$.



The Laplace Transform and the IVP (Sect. 6.2).

- ▶ Solving differential equations using $\mathcal{L}[\]$.
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 - ▶ **First, second, higher order equations.**
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First, second, higher order equations.

Example

Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

Solution: Compute the $\mathcal{L}[\]$ of the equation,

$$\mathcal{L}[y^{(4)}] - 4\mathcal{L}[y] = 0.$$

$$[s^4 \mathcal{L}[y] - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)] - 4\mathcal{L}[y] = 0.$$

$$[s^4 \mathcal{L}[y] - s^3 + 2s] - 4\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\mathcal{L}[y] = s^3 - 2s,$$

We obtain, $\mathcal{L}[y] = \frac{s^3 - 2s}{(s^4 - 4)}$.

First, second, higher order equations.

Example

Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \frac{s^3 - 2s}{(s^4 - 4)}$.

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{s}{(s^2 + 2)}.$$

The last expression is in the table of Laplace Transforms,

$$\mathcal{L}[y] = \frac{s}{(s^2 + [\sqrt{2}]^2)} = \mathcal{L}[\cos(\sqrt{2} t)].$$

We conclude that $y(t) = \cos(\sqrt{2} t)$.

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Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Compute the Laplace transform of the equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3 \sin(2t)].$$

The right-hand side above can be expressed as follows,

$$\mathcal{L}[3 \sin(2t)] = 3 \mathcal{L}[\sin(2t)] = 3 \frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}.$$

Introduce this source term in the differential equation,

$$\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y''] - 4\mathcal{L}[y'] + 4\mathcal{L}[y] = \frac{6}{s^2 + 4}$.

Derivatives are transformed into power functions,

$$\left[s^2 \mathcal{L}[y] - s y(0) - y'(0) \right] - 4 \left[s \mathcal{L}[y] - y(0) \right] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

Rewrite the above equation,

$$(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0) + \frac{6}{s^2 + 4}.$$

Introduce the initial conditions,

$$(s^2 - 4s + 4) \mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4) \mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}$.

Therefore, $\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4s + 4)(s^2 + 4)}$.

From an Example above: $s^2 - 4s + 4 = (s - 2)^2$,

$$\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2} + \frac{6}{(s - 2)^2(s^2 + 4)}.$$

From an Example above we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \frac{6}{(s-2)^2(s^2+4)}$.

Use Partial fractions to simplify the last term above.

Find constants a, b, c, d , such that

$$\frac{6}{(s-2)^2(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{s-2} + \frac{d}{(s-2)^2}$$

$$\frac{6}{(s-2)^2(s^2+4)} = \frac{(as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)}{(s^2+4)(s-2)^2}$$

$$6 = (as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4).$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: $6 = (as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)$.

$$6 = (as+b)(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4)$$

$$6 = a(s^3 - 4s^2 + 4s) + b(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4).$$

$$6 = (a+c)s^3 + (-4a+b-2c+d)s^2 + (4a-4b+4c)s + (4b-8c+4d).$$

We obtain the system

$$\begin{aligned} a + c &= 0, & -4a + b - 2c + d &= 0, \\ 4a - 4b + 4c &= 0, & 4b - 8c + 4d &= 6. \end{aligned}$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: The solution for this linear system is

$$a = \frac{3}{8}, \quad b = 0, \quad c = -\frac{3}{8}, \quad d = \frac{3}{4}.$$

$$\frac{6}{(s-2)^2(s^2+4)} = \frac{3}{8} \frac{s}{s^2+4} - \frac{3}{8} \frac{1}{(s-2)} + \frac{3}{4} \frac{1}{(s-2)^2}.$$

Use the table of Laplace Transforms

$$\frac{6}{(s-2)^2(s^2+4)} = \frac{3}{8} \mathcal{L}[\cos(2t)] - \frac{3}{8} \mathcal{L}[e^{2t}] + \frac{3}{4} \mathcal{L}[te^{2t}].$$

$$\frac{6}{(s-2)^2(s^2+4)} = \mathcal{L}\left[\frac{3}{8} \cos(2t) - \frac{3}{8} e^{2t} + \frac{3}{4} te^{2t}\right].$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Summary: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \frac{6}{(s-2)^2(s^2+4)},$

$$\frac{6}{(s-2)^2(s^2+4)} = \mathcal{L}\left[\frac{3}{8} \cos(2t) - \frac{3}{8} e^{2t} + \frac{3}{4} te^{2t}\right].$$

$$\mathcal{L}[y(t)] = \mathcal{L}\left[(1-t)e^{2t} + \frac{3}{8}(-1+2t)e^{2t} + \frac{3}{8} \cos(2t)\right].$$

We conclude that

$$y(t) = (1-t)e^{2t} + \frac{3}{8}(2t-1)e^{2t} + \frac{3}{8} \cos(2t). \quad \triangleleft$$