The Laplace Transform (Sect. 6.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.

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The definition of the Laplace Transform.

Definition

The function $F : D_F \to \mathbb{R}$ is the *Laplace transform* of a function $f : [0, \infty) \to \mathbb{R}$ iff for all $s \in D_F$ holds,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

where $D_F \subset \mathbb{R}$ is the set where the integral converges.

Remark: The domain D_F of F depends on the function f.

Notation: We often denote: $F(s) = \mathcal{L}[f(t)]$.

- This notation $\mathcal{L}[\ ]$ emphasizes that the Laplace transform defines a map from a set of functions into a set of functions.
- Functions are denoted as $t \mapsto f(t)$.
- The Laplace transform is also a function: $f \mapsto \mathcal{L}[f]$.

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Review: Improper integrals.

Recall: Improper integral are defined as a limit.

$$\int_{t_0}^{\infty} g(t)\,dt = \lim_{N\to\infty} \int_{t_0}^{N} g(t)\,dt.$$

- The integral converges iff the limit exists.
- The integral diverges iff the limit does not exist.

Example

Compute the improper integral $\displaystyle\int_0^{\infty} e^{-at}\,dt$, with $a > 0$.

Solution: $\displaystyle\int_0^{\infty} e^{-at}\,dt = \lim_{N\to\infty} \int_0^{N} e^{-at}\,dt = \lim_{N\to\infty} -\frac{1}{a}\Big(e^{-aN} - 1\Big).$

Since $\displaystyle\lim_{N\to\infty} e^{-aN} = 0$ for $a > 0$, we conclude $\displaystyle\int_0^{\infty} e^{-at}\,dt = \frac{1}{a}.$ $\lhd$

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Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[1]$.

Solution: We have to find the Laplace Transform of $f(t) = 1$. Following the definition we obtain,

$$\mathcal{L}[1] = \int_0^{\infty} e^{-st}\, 1\, dt = \int_0^{\infty} e^{-st}\, dt$$

But $\displaystyle\int_0^{\infty} e^{-at}\, dt = \frac{1}{a}$ for $a > 0$, and diverges for $a \leqslant 0$.

Therefore $\mathcal{L}[1] = \dfrac{1}{s}$, for $s > 0$, and $\mathcal{L}[1]$ does not exists for $s \leqslant 0$.

In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \to \mathbb{R}$ given by

$$f(t) = 1, \qquad F(s) = \frac{1}{s}, \qquad D_F = (0, \infty). \qquad \triangleleft$$

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

Solution: Following the definition of Laplace Transform,

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-st} e^{at}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt.$$

We have seen that the improper integral is given by

$$\int_0^{\infty} e^{-(s-a)}\, dt = \frac{1}{(s-a)} \quad \text{for} \quad (s-a) > 0.$$

We conclude that $\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$ for $s > a$. In other words,

$$f(t) = e^{at}, \qquad F(s) = \frac{1}{(s-a)}, \qquad s > a. \qquad \triangleleft$$

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: In this case we need to compute

$$\mathcal{L}[\sin(at)] = \lim_{N\to\infty} \int_0^N e^{-st} \sin(at)\, dt.$$

Integrating by parts twice it is not difficult to obtain:

$$\int_0^N e^{-st} \sin(at)\, dt =$$

$$-\frac{1}{s}\left[e^{-st}\sin(at)\right]\Big|_0^N - \frac{a}{s^2}\left[e^{-st}\cos(at)\right]\Big|_0^N - \frac{a^2}{s^2}\int_0^N e^{-st}\sin(at)\, dt.$$

This identity implies

$$\left(1 + \frac{a^2}{s^2}\right)\int_0^N e^{-st}\sin(at)\, dt = -\frac{1}{s}\left[e^{-st}\sin(at)\right]\Big|_0^N - \frac{a}{s^2}\left[e^{-st}\cos(at)\right]\Big|_0^N.$$

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: Recall the identity:

$$\left(1 + \frac{a^2}{s^2}\right)\int_0^N e^{-st}\sin(at)\, dt = -\frac{1}{s}\left[e^{-st}\sin(at)\right]\Big|_0^N - \frac{a}{s^2}\left[e^{-st}\cos(at)\right]\Big|_0^N.$$

Hence, it is not difficult to see that

$$\left(\frac{s^2 + a^2}{s^2}\right)\int_0^\infty e^{-st}\sin(at)\, dt = \frac{a}{s^2}, \qquad s > 0,$$

which is equivalent to

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \qquad s > 0. \qquad \triangleleft$$

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A table of Laplace Transforms.

$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0$,
$f(t) = e^{at}$	$F(s) = \dfrac{1}{s-a}$	$s > \max\{a, 0\}$,
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$,
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0$,
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0$,
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > 0$,
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > 0$,
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > \max\{a, 0\}$,
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > \max\{a, 0\}$.

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Properties of the Laplace Transform.

Theorem (Sufficient conditions)

If the function $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and there exist positive constants k and a such that

$$|f(t)| \leqslant k\, e^{at},$$

then the Laplace Transform of f exists for all $s > a$.

Theorem (Linear combination)

If the $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are well-defined and a, b are constants, then

$$\mathcal{L}[af + bg] = a\,\mathcal{L}[f] + b\,\mathcal{L}[g].$$

Proof: Integration is a linear operation:

$$\int [a\,f(t) + b\,g(t)]\,dt = a \int f(t)\,dt + b \int g(t)\,dt.$$

Properties of the Laplace Transform.

Theorem (Derivatives)

If the $\mathcal{L}[f]$ and $\mathcal{L}[f']$ are well-defined, then holds,

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0). \tag{1}$$

Furthermore, if $\mathcal{L}[f'']$ is well-defined, then it also holds

$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0). \tag{2}$$

Proof of Eq (2): Use Eq. (1) twice:

$$\mathcal{L}[f''] = \mathcal{L}[(f')'] = s\mathcal{L}[(f')] - f'(0) = s\big(s\mathcal{L}[f] - f(0)\big) - f'(0),$$

that is,

$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0).$$

Properties of the Laplace Transform.

Proof of Eq (1): Recall the definition of the Laplace Transform,

$$\mathcal{L}[f'] = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{n\to\infty} \int_0^n e^{-st} f'(t)\,dt$$

Integrating by parts,

$$\lim_{n\to\infty} \int_0^n e^{-st} f'(t)\,dt = \lim_{n\to\infty} \Big[\Big(e^{-st} f(t)\Big)\Big|_0^n - \int_0^n (-s)e^{-st} f(t)\,dt\Big]$$

$$\mathcal{L}[f'] = \lim_{n\to\infty}\Big[e^{-sn} f(n) - f(0)\Big] + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + s\,\mathcal{L}[f],$$

where we used that $\lim_{n\to\infty} e^{-sn} f(n) = 0$ for s big enough, and we also used that $\mathcal{L}[f]$ is well-defined.

We then conclude that $\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0)$.

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Laplace Transform and differential equations.

Remark: Laplace Transforms can be used to find solutions to differential equations with constant coefficients.

Idea of the method:

$$\mathcal{L}\begin{bmatrix}\text{Differential Eq.}\\ \text{for } y(t).\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic Eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(2)}$$

$$\xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \text{Algebraic Eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y(t).\\ \text{(Using the table.)}\end{matrix}$$

Laplace Transform and differential equations.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \qquad y(0) = 3.$$

Solution: We know the solution: $y(t) = 3e^{-2t}$.

(1): Compute the Laplace transform of the differential equation,

$$\mathcal{L}[y' + 2y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y' + 2y] = 0.$$

Find an algebraic equation for $\mathcal{L}[y]$. Recall linearity:

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = 0.$$

Also recall the property: $\mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0)$, that is,

$$\Big[s\,\mathcal{L}[y] - y(0)\Big] + 2\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s+2)\mathcal{L}[y] = y(0).$$

Laplace Transform and differential equations.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \qquad y(0) = 3.$$

Solution: Recall: $(s+2)\mathcal{L}[y] = y(0)$.

(2): Solve the algebraic equation for $\mathcal{L}[y]$.

$$\mathcal{L}[y] = \frac{y(0)}{s+2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s+2}.$$

(3): Transform back to $y(t)$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \;\Rightarrow\; \frac{3}{s+2} = 3\mathcal{L}[e^{-2t}] \;\Rightarrow\; \frac{3}{s+2} = \mathcal{L}[3e^{-2t}].$$

Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \quad \Rightarrow \quad y(t) = 3e^{-2t}$. $\triangleleft$