Non-homogeneous equations (Sect. 3.5).

- We study: $y'' + a_1\, y' + a_0\, y = b(t)$.
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
- The guessing solution table.

Operator notation and preliminary results.

Notation: Given functions p, q, denote

$$L(y) = y'' + p(t)\, y' + q(t)\, y.$$

Therefore, the differential equation

$$y'' + p(t)\, y' + q(t)\, y = f(t)$$

can be written as

$$L(y) = f.$$

The homogeneous equation can be written as

$$L(y) = 0.$$

The function L acting on a function y is called an operator.

Operator notation and preliminary results.

Remark: The operator L is a linear function of y.

Theorem

For every continuously differentiable functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ and every c_1, $c_2 \in \mathbb{R}$ holds that

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$

Proof:

$$L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)'' + p(t)\,(c_1 y_1 + c_2 y_2)' + q(t)\,(c_1 y_1 + c_2 y_2)$$

$$\begin{aligned} L(c_1 y_1 + c_2 y_2) = &\big(c_1 y_1'' + p(t)\, c_1 y_1' + q(t)\, c_1 y_1\big) \\ &+ \big(c_2 y_2'' + p(t)\, c_2 y_2' + q(t)\, c_2 y_2\big) \end{aligned}$$

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$ □

Operator notation and preliminary results.

Theorem

Given functions p, q, f, let $L(y) = y'' + p(t)\, y' + q(t)\, y$.
If the functions y_1 and y_2 are fundamental solutions of the homogeneous equation

$$L(y) = 0,$$

and y_p is any solution of the non-homogeneous equation

$$L(y_p) = f, \tag{1}$$

then any other solution y of the non-homogeneous equation above is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \tag{2}$$

where c_1, $c_2 \in \mathbb{R}$.

Notation: The expression for y in Eq. (2) is called the general solution of the non-homogeneous Eq. (1).

Operator notation and preliminary results.

Theorem

Given functions p, q, let $L(y) = y'' + p(t)\, y' + q(t)\, y$.
If the function f can be written as $f(t) = f_1(t) + \cdots + f_n(t)$, with $n \geqslant 1$, and if there exist functions $y_{p_1}, \cdots, y_{p_n}$ such that

$$L(y_{p_i}) = f_i, \qquad i = 1, \cdots, n,$$

then the function $y_p = y_{p_1} + \cdots + y_{p_n}$ satisfies the non-homogeneous equation

$$L(y_p) = f.$$

Non-homogeneous equations (Sect. 3.5).

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Summary of the undetermined coefficients method.

Problem: Given a constant coefficients linear operator $L(y) = y'' + a_1 y' + a_0 y$, with a_1, $a_2 \in \mathbb{R}$, find every solution of the non-homogeneous differential equation

$$L(y) = f.$$

Remarks:

- The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.
- It consists in guessing the solution y_p of the non-homogeneous equation
$$L(y_p) = f,$$
for particularly simple source functions f.

Summary of the undetermined coefficients method.

Summary:

(1) Find the general solution of the homogeneous equation $L(y_h) = 0$.

(2) If f has the form $f = f_1 + \cdots + f_n$, with $n \geqslant 1$, then look for solutions y_{p_i}, with $i = 1, \cdots, n$ to the equations

$$L(y_{p_i}) = f_i.$$

Once the functions y_{p_i} are found, then construct

$$y_p = y_{p_1} + \cdots + y_{p_n}.$$

(3) Given the source functions f_i, guess the solutions functions y_{p_i} following the Table below.

Summary of the undetermined coefficients method.

Summary (cont.):

$f_i(t)$ (K, m, a, b, given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1\cos(bt) + k_2\sin(bt)$
$K\sin(bt)$	$k_1\cos(bt) + k_2\sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at}\cos(bt)$	$e^{at}\big[k_1\cos(bt) + k_2\sin(bt)\big]$
$KKe^{at}\sin(bt)$	$e^{at}\big[k_1\cos(bt) + k_2\sin(bt)\big]$
$Kt^m\cos(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1\cos(bt) + a_2\sin(bt)\big]$
$Kt^m\sin(bt)$	$\big(k_m t^m + \cdots + k_0\big)\big[a_1\cos(bt) + a_2\sin(bt)\big]$

Summary of the undetermined coefficients method.

Summary (cont.):

(4) If any guessed function y_{p_i} satisfies the homogeneous equation $L(y_{p_i}) = 0$, then change the guess to the function

$$t^s y_{p_i}, \quad \text{with} \quad s \geqslant 1,$$

and s sufficiently large such that $L(t^s y_{p_i}) \neq 0$.

(5) Impose the equation $L(y_{p_i}) = f_i$ to find the undetermined constants $k_1, \cdots, k_m$, for the appropriate m, given in the table above.

(6) The general solution to the original differential equation $L(y) = f$ is then given by

$$y(t) = y_h(t) + y_{p_1} + \cdots + y_{p_n}.$$

Non-homogeneous equations (Sect. 3.5).

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Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution: Notice: $L(y) = y'' - 3y' - 4y$ and $f(t) = 3e^{2t}$.

(1) Find all solutions y_h to the homogeneous equation $L(y_h) = 0$.
The characteristic equation is

$$r^2 - 3r - 4 = 0 \quad \Rightarrow \quad \begin{cases} r_1 = 4, \\ r_2 = -1. \end{cases}$$

$$y_h(t) = c_1\, e^{4t} + c_2\, e^{-t}.$$

(2) Trivial in our case. The source function $f(t) = 3e^{2t}$ cannot be simplified into a sum of simpler functions.

(3) Table says: For $f(t) = 3e^{2t}$ guess $y_p(t) = k\, e^{2t}$

Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution: Recall: $y_p(t) = k\, e^{2t}$. We need to find k.

(4) Trivial here, since $L(y_p) \neq 0$, we do not modify our guess. (Recall: $L(y_h) = 0$ iff $y_h(t) = c_1\, e^{4t} + c_2\, e^{-t}$.)

(5) Introduce y_p into $L(y_p) = f$ and find k.

$$(2^2 - 6 - 4)ke^{2t} = 3e^{2t} \quad \Rightarrow \quad -6k = 3 \quad \Rightarrow \quad k = -\frac{1}{2}.$$

We have obtained that $y_p(t) = -\dfrac{1}{2}\, e^{2t}$.

(6) The general solution to the inhomogeneous equation is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}\, e^{2t}.$$

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Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{4t}.$$

Solution: We know that the general solution to homogeneous equation is $y_h(t) = c_1 e^{4t} + c_2 e^{-t}$.

Following the table we guess y_p as $y_p = k\, e^{4t}$.

However, this guess satisfies $L(y_p) = 0$.

So we modify the guess to $y_p = kt\, e^{4t}$.

Introduce the guess into $L(y_p) = f$. We need to compute

$$y_p' = k\, e^{4t} + 4kt\, e^{4t}, \qquad y_p'' = 8k\, e^{4t} + 16kt\, e^{4t}.$$

Using the method in few examples.

Example

Find all solutions to the non-homogeneous equation

$$y'' - 3y' - 4y = 3e^{4t}.$$

Solution: Recall:

$$y_p = kt\,e^{4t}, \quad y_p' = k\,e^{4t} + 4kt\,e^{4t}, \quad y_p'' = 8k\,e^{4t} + 16kt\,e^{4t}.$$

$$\big[(8k + 16kt) - 3(k + 4kt) - 4kt\big]\,e^{4t} = 3e^{4t}.$$

$$\big[(8+16t)-3(1+4t)-4t\big]\,k = 3 \quad \Rightarrow \quad \big[5+(16-12-4)\,t\big]\,k = 3$$

We obtain that $k = \frac{3}{5}$. Therefore, $y_p(t) = \frac{3}{5}\,t\,e^{4t}$, and

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{3}{5}\,t\,e^{4t}.$$

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Using the method in few examples.

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: We know that the general solution to homogeneous equation is $y(t) = c_1 e^{4t} + c_2 e^{-t}$.

Following the table: Since $f = 2\sin(t)$, then we guess

$$y_p = k_1 \sin(t) + k_2 \cos(t).$$

This guess satisfies $L(y_p) \neq 0$.

Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$

$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

Using the method in few examples.

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall:

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$
$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

$$(-5k_1 + 3k_2)\sin(t) + (-3k_1 - 5k_2)\cos(t) = 2\sin(t).$$

This equation holds for all $t \in \mathbb{R}$. In particular, at $t = \frac{\pi}{2}$, $t = 0$.

$$\left.\begin{aligned} -5k_1 + 3k_2 &= 2, \\ -3k_1 - 5k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} k_1 &= -\frac{5}{17}, \\ k_2 &= \frac{3}{17}. \end{aligned}\right.$$

Using the method in few examples.

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall: $k_1 = -\frac{5}{17}$ and $k_2 = \frac{3}{17}$.

So the particular solution to the inhomogeneous equation is

$$y_p(t) = \frac{1}{17}\big[-5\sin(t) + 3\cos(t)\big].$$

The general solution is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17}\big[-5\sin(t) + 3\cos(t)\big]. \qquad \triangleleft$$

Using the method in few examples.

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t).$$

Solution: We know that the general solution y is given by

$$y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t),$$

where $y_h(t) = c_1 e^{4t} + c_2 e^{2t}$, $L(y_{p_1}) = 3e^{2t}$, and $L(y_{p_2}) = 2\sin(t)$.
We have just found out that

$$y_p(t) = -\frac{1}{2}\, e^{2t}, \qquad y_{p_2}(t) = \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right].$$

We conclude that

$$y(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{1}{2}\, e^{2t} + \frac{1}{17}\left[-5\sin(t) + 3\cos(t)\right]. \quad \triangleleft$$

Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess
$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$
- For $y'' - 3y' - 4y = 2t^2\, e^{3t}$, guess
$$y_p(t) = \left(k_0 + k_1 t + k_2 t^2\right) e^{3t}.$$
- For $y'' - 3y' - 4y = 3t\, \sin(t)$, guess
$$y_p(t) = (1 + k_1 t)\left[k_2 \sin(t) + k_3 \cos(t)\right].$$

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The guessing solution table.

Guessing Solution Table.

$f_i(t)$ (K, m, a, b, given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K\cos(bt)$	$k_1\cos(bt) + k_2\sin(bt)$
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