Second order linear homogeneous ODE (Sect. 3.3).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.

Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Definition
Any two solutions y_1, y_2 of the homogeneous equation

$$y'' + a_1(t)y' + a_0(t)y = 0,$$

are called *fundamental solutions* iff the functions y_1, y_2 are linearly independent, that is, iff $W_{y_1 y_2} \neq 0$.

Remark: Fundamental solutions are not unique.

Definition
Given any two fundamental solutions y_1, y_2, and arbitrary constants c_1, c_2, the function

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

is called the *general solution* of the differential equation above.

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Theorem (Constant coefficients)

Given real constants a_1, a_0, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1 y' + a_0 y = 0. \tag{1}$$

Let r_+, r_- be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let c_0, c_1 be arbitrary constants. Then, any solution of Eq. (1) belongs to only one of the following cases:

(a) *If $r_+ \neq r_-$, the general solution is $y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$.*

(b) *If $r_+ = r_- \in \mathbb{R}$, the general solution is $y(t) = (c_1 + c_2 t) e^{r_+ t}$.*

Furthermore, given real constants t_0, y_1 and y_2, there is a unique solution to the initial value problem given by Eq. (1) and the initial conditions

$$y(t_0) = y_1, \qquad y'(t_0) = y_2.$$

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example

Find the general solution of the equation $y'' - y' - 6y = 0$.

Solution: Since solutions have the form e^{rt}, we need to find the roots of the characteristic polynomial $p(r) = r^2 - r - 6$, that is,

$$r_\pm = \frac{1}{2}\left(1 \pm \sqrt{1+24}\right) = \frac{1}{2}(1 \pm 5) \quad \Rightarrow \quad r_+ = 3, \quad r_- = -2.$$

So, $r_\pm$ are real-valued. A fundamental solution set is formed by

$$y_1(t) = e^{3t}, \qquad y_2(t) = e^{-2t}.$$

The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions, that is,

$$y(t) = c_1\, e^{3t} + c_2\, e^{-2t}, \qquad c_1,\, c_2 \in \mathbb{R}. \qquad \lhd$$

Remark: Since c_1, $c_2 \in \mathbb{R}$, then y is real-valued.

Second order linear homogeneous ODE.

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Two main sets of fundamental solutions.

Theorem (Complex roots)

If the constants a_1, $a_0 \in \mathbb{R}$ satisfy that $a_1^2 - 4a_0 < 0$, then the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$ of the equation

$$y'' + a_1\, y' + a_0\, y = 0 \tag{2}$$

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{a_1}{2}, \qquad \beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}.$$

Furthermore, a fundamental set of solutions to Eq. (2) is

$$\tilde{y}_1(t) = e^{(\alpha + i\beta)t}, \qquad \tilde{y}_2(t) = e^{(\alpha - i\beta)t},$$

while another fundamental set of solutions to Eq. (2) is

$$y_1(t) = e^{\alpha t} \cos(\beta t), \qquad y_2(t) = e^{\alpha t} \sin(\beta t).$$

Two main sets of fundamental solutions.

Example

Find the general solution of the equation $y'' - 2y' + 6y = 0$.

Solution: We first find the roots of the characteristic polynomial,

$$r^2 - 2r + 6 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(2 \pm \sqrt{4 - 24}\big) \quad \Rightarrow \quad r_\pm = 1 \pm i\sqrt{5}.$$

A fundamental solution set is

$$\tilde{y}_1(t) = e^{(1+i\sqrt{5})\,t}, \qquad \tilde{y}_2(t) = e^{(1-i\sqrt{5})\,t}.$$

These are complex-valued functions. The general solution is

$$y(t) = \tilde{c}_1\, e^{(1+i\sqrt{5})\,t} + \tilde{c}_2\, e^{(1-i\sqrt{5})\,t}, \qquad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}. \qquad \triangleleft$$

Two main sets of fundamental solutions.

Remark:

- The solutions found above include real-valued and complex-valued solutions.
- Since the differential equation is real-valued, it is usually important in applications to obtain the most general real-valued solution. (See RLC circuit below.)
- In the expression above it is difficult to take apart real-valued solutions from complex-valued solutions.
- In other words: It is not simple to see what values of $\tilde{c}_1$ and $\tilde{c}_2$ make the general solution above to be real-valued.
- One way to find the real-valued general solution is to find real-valued fundamental solutions.

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A real-valued fundamental and general solutions.

Example

Find the real-valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$

Solution: Recall: $y(t) = \tilde{c}_1 e^{(1+i\sqrt{5})\,t} + \tilde{c}_2 e^{(1-i\sqrt{5})\,t}$, $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}$.

The Theorem above says that a real-valued fundamental set is

$$y_1(t) = e^t \cos(\sqrt{5}\,t), \qquad y_2(t) = e^t \sin(\sqrt{5}\,t).$$

Hence, the complex-valued general solution can also be written as

$$y(t) = \big[c_1 \cos(\sqrt{5}\,t) + c_2 \sin(\sqrt{5}\,t)\big]\, e^t, \qquad c_1, c_2 \in \mathbb{C}.$$

The real-valued general solution is simple to obtain:

$$y(t) = \big[c_1 \cos(\sqrt{5}\,t) + c_2 \sin(\sqrt{5}\,t)\big]\, e^t, \qquad c_1, c_2 \in \mathbb{R}.$$

We just restricted the coefficients c_1, c_2 to be real-valued. $\lhd$

A real-valued fundamental and general solutions.

Example

Show that $y_1(t) = e^t \cos(\sqrt{5}\,t)$ and $y_2(t) = e^t \sin(\sqrt{5}\,t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

Solution: We start with the complex-valued fundamental solutions,

$$\tilde{y}_1(t) = e^{(1+i\sqrt{5})\,t}, \qquad \tilde{y}_2(t) = e^{(1-i\sqrt{5})\,t}.$$

Any linear combination of these functions is solution of the differential equation. In particular,

$$y_1(t) = \frac{1}{2}\big[\tilde{y}_1(t) + \tilde{y}_2(t)\big], \quad y_2(t) = \frac{1}{2i}\big[\tilde{y}_1(t) - \tilde{y}_2(t)\big].$$

Now, recalling $e^{(1\pm i\sqrt{5})\,t} = e^t e^{\pm i\sqrt{5}\,t}$

$$y_1(t) = \frac{1}{2}\big[e^t\, e^{i\sqrt{5}t} + e^t\, e^{-i\sqrt{5}t}\big], \quad y_2(t) = \frac{1}{2i}\big[e^t\, e^{i\sqrt{5}t} - e^t\, e^{-i\sqrt{5}t}\big],$$

A real-valued fundamental and general solutions.

Example

Show that $y_1(t) = e^t \cos(\sqrt{5}\,t)$ and $y_2(t) = e^t \sin(\sqrt{5}\,t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

Solution: $y_1 = \dfrac{e^t}{2}\big[e^{i\sqrt{5}t} + e^{-i\sqrt{5}t}\big]$, $y_2 = \dfrac{e^t}{2i}\big[e^{i\sqrt{5}t} - e^{-i\sqrt{5}t}\big]$.

The Euler formula and its complex-conjugate formula

$$e^{i\sqrt{5}t} = \big[\cos(\sqrt{5}\,t) + i\sin(\sqrt{5}\,t)\big],$$

$$e^{-i\sqrt{5}\,t} = \big[\cos(\sqrt{5}\,t) - i\sin(\sqrt{5}\,t)\big],$$

imply the inverse relations

$$e^{i\sqrt{5}t} + e^{-i\sqrt{5}t} = 2\cos(\sqrt{5}t), \quad e^{i\sqrt{5}t} - e^{-i\sqrt{5}t} = 2i\sin(\sqrt{5}t).$$

So functions y_1 and y_2 can be written as

$$y_1(t) = e^t\cos(\sqrt{5}\,t), \qquad y_2(t) = e^t\sin(\sqrt{5}\,t).$$

A real-valued fundamental and general solutions.

Example

Show that $y_1(t) = e^t \cos(\sqrt{5}\, t)$ and $y_2(t) = e^t \sin(\sqrt{5}\, t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

Solution: $y_1(t) = e^t \cos(\sqrt{5}\, t)$, $y_2(t) = e^t \sin(\sqrt{5}\, t)$.

Summary:

- These functions are solutions of the differential equation.
- They are not proportional to each other, Hence li.
- Therefore, y_1, y_2 form a fundamental set.
- The general solution of the equation is

$$y(t) = \left[c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)\right] e^t.$$

- y is real-valued for $c_1, c_2 \in \mathbb{R}$.
- y is complex-valued for $c_1, c_2 \in \mathbb{C}$.

A real-valued fundamental and general solutions.

Remark:

- The proof of the Theorem follow exactly the same ideas given in the example above.
- One has to replace the roots of the characteristic polynomial

$$1 + i\sqrt{5} \quad \to \quad \alpha + i\beta, \qquad 1 - i\sqrt{5} \quad \to \quad \alpha - i\beta.$$

- The real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t} \cos(\beta t), \qquad y_2(t) = e^{\alpha t} \sin(\beta t).$$

A real-valued fundamental and general solutions.

Example

Find real-valued fundamental solutions to the equation

$$y'' + 2\,y' + 6\,y = 0.$$

Solution:

The roots of the characteristic polynomial $p(r) = r^2 + 2r + 6$ are

$$r_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4-24}\big] = \frac{1}{2}\big[-2 \pm \sqrt{-20}\big] \;\Rightarrow\; r_\pm = -1 \pm i\sqrt{5}.$$

These are complex-valued roots, with

$$\alpha = -1, \qquad \beta = \sqrt{5}.$$

Real-valued fundamental solutions are

$$y_1(t) = e^{-t}\,\cos(\sqrt{5}\,t), \qquad y_2(t) = e^{-t}\,\sin(\sqrt{5}\,t). \qquad \triangleleft$$

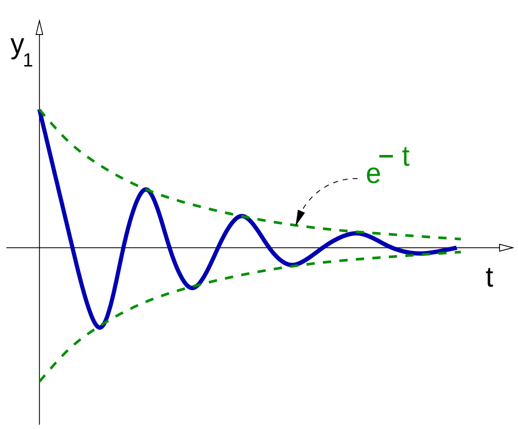
A real-valued fundamental and general solutions.

Example

Find real-valued fundamental solutions to the equation

$$y'' + 2\,y' + 6\,y = 0.$$

Solution: $y_1(t) = e^{-t}\,\cos(\sqrt{5}\,t)$, $y_2(t) = e^{-t}\,\sin(\sqrt{5}\,t)$.



Differential equations like the one in this example describe physical processes related to damped oscillations. For example pendulums with friction.

A real-valued fundamental and general solutions.

Example

Find the real-valued general solution of $y'' + 5\,y = 0$.

Solution: The characteristic polynomial is $p(r) = r^2 + 5$.

Its roots are $r_{\pm} = \pm\sqrt{5}\,i$. This is the case $\alpha = 0$, and $\beta = \sqrt{5}$.

Real-valued fundamental solutions are

$$y_1(t) = \cos(\sqrt{5}\,t), \qquad y_2(t) = \sin(\sqrt{5}\,t).$$

The real-valued general solution is

$$y(t) = c_1 \cos(\sqrt{5}\,t) + c_2 \sin(\sqrt{5}\,t), \qquad c_1, c_2 \in \mathbb{R}. \qquad \triangleleft$$

Remark: Equations like the one in this example describe oscillatory physical processes without dissipation.

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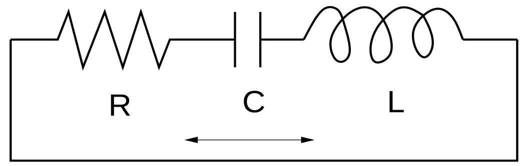
Application: The RLC circuit.

Consider an electric circuit with resistance R, non-zero capacitor C, and non-zero inductance L, as in the figure.



I (t) : electric current.

The electric current flowing in such circuit satisfies:

$$L\,I'(t) + R\,I(t) + \frac{1}{C}\int_{t_0}^{t} I(s)\,ds = 0.$$

Derivate both sides above: $L\,I''(t) + R\,I'(t) + \dfrac{1}{C}\,I(t) = 0.$

Divide by L: $I''(t) + 2\left(\dfrac{R}{2L}\right) I'(t) + \dfrac{1}{LC}\,I(t) = 0.$

Introduce $\alpha = \dfrac{R}{2L}$ and $\omega = \dfrac{1}{\sqrt{LC}}$, then $I'' + 2\alpha\,I' + \omega^2\,I = 0.$

Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\,I' + \omega^2\,I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2}\left[-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}\right] \quad \Rightarrow \quad r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$

Case (a) $R = 0$. This implies $\alpha = 0$, so $r_\pm = \pm i\omega$. Therefore,

$$I_1(t) = \cos(\omega t), \qquad I_2(t) = \sin(\omega t).$$

Remark: When the circuit has no resistance, the current oscillates without dissipation.

Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

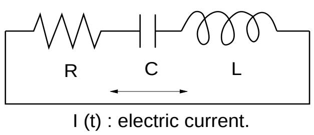
Solution: Recall: $r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

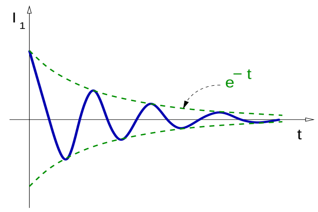
Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \quad \Leftrightarrow \quad \frac{R^2}{4L^2} < \frac{1}{LC} \quad \Leftrightarrow \quad \alpha^2 < \omega^2.$$

Therefore, $r_\pm = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos\big(\sqrt{\omega^2 - \alpha^2}\, t\big), \quad I_2(t) = e^{-\alpha t} \sin\big(\sqrt{\omega^2 - \alpha^2}\, t\big).$$





The resistance R damps the current oscillations.