Review 2 for Exam 1.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Linear equations (2.1).
  - Separable equations (2.2).
  - Homogeneous equations (2.2).
  - Modeling using differential equations (2.3).
  - Non-linear equations (2.4).
  - Bernoulli equation (2.4).
  - Autonomous systems (2.5).
  - Exact equations (2.6).
  - Exact equations with integrating factors (2.6).

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Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\Big(x^3 e^y + \frac{x}{y}\Big)\, y' + (2x^2 e^y + 1) = 0.$$

Solution: We first verify if the equation is not exact.

$$N = \Big(x^3 e^y + \frac{x}{y}\Big) \quad \Rightarrow \quad \partial_x N = 3x^2 e^y + \frac{1}{y}.$$

$$M = (2x^2 e^y + 1) = 0 \quad \Rightarrow \quad \partial_y M = 2x^2 e^y.$$

So the equation is not exact. We now compute

$$\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \Big(3x^2 e^y + \dfrac{1}{y}\Big)}{\Big(x^3 e^y + \dfrac{x}{y}\Big)} = \frac{-x^2 e^y - \dfrac{1}{y}}{x\Big(x^2 e^y + \dfrac{1}{y}\Big)} = -\frac{1}{x}.$$

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Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\Big(x^3 e^y + \frac{x}{y}\Big)\, y' + (2x^2 e^y + 1) = 0.$$

Solution: Recall: $\dfrac{\partial_y M - \partial_x N}{N} = -\dfrac{1}{x}$. Therefore,

$$\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x} \quad \Rightarrow \quad \ln(\mu) = -\ln(x) = \ln\Big(\frac{1}{x}\Big) \quad \Rightarrow \quad \mu(x) = \frac{1}{x}.$$

So the equation $\Big(x^2 e^y + \dfrac{1}{y}\Big)\, y' + \Big(2xe^y + \dfrac{1}{x}\Big) = 0$ is exact. Indeed,

$$\left.\begin{array}{l} \tilde{N} = \Big(x^2 e^y + \dfrac{1}{y}\Big) \quad \Rightarrow \quad \partial_x \tilde{N} = 2xe^y, \\ \tilde{M} = \Big(2xe^y + \dfrac{1}{x}\Big) \quad \Rightarrow \quad \partial_y \tilde{M} = 2xe^y, \end{array}\right\} \quad \Rightarrow \quad \partial_x \tilde{N} = \partial_y \tilde{M}.$$

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Example

Find every solution y of the equation

$$\Big(x^2 e^y + \frac{1}{y}\Big)\, y' + \Big(2x\, e^y + \frac{1}{x}\Big) = 0.$$

Solution: The equation is exact. We need to find the potential function ψ.

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

From the first equation we get:

$$\partial_y \psi = x^2 e^y + \frac{1}{y} \quad \Rightarrow \quad \psi = x^2 e^y + \ln(y) + g(x).$$

Introduce the expression for ψ in the equation $\partial_x \psi = M$, that is,

$$2xe^y + g'(x) = \partial_x \psi = M = 2x\, e^y + \frac{1}{x} \quad \Rightarrow \quad g'(x) = \frac{1}{x}.$$

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Example

Find every solution y of the equation

$$\Big(x^2 e^y + \frac{1}{y}\Big)\, y' + \Big(2x\, e^y + \frac{1}{x}\Big) = 0.$$

Solution: Recall: $g'(x) = \dfrac{1}{x}$. Therefore $g(x) = \ln(x)$.

The potential function is $\psi = x^2 e^y + \ln(y) + \ln(x)$.

The solution y satisfies $x^2 e^{y(x)} + \ln(y(x)) + \ln(x) = c$. $\triangleleft$

Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.

$$2xe^y + x^2 e^y y' + \frac{1}{y}\, y' + \frac{1}{x} = 0.$$

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Example

Find every solution of the initial value problem

$$y' = 4x(y + \sqrt{y}), \qquad y(0) = 4.$$

Solution: The equation is: Not linear.

It is a Bernoulli equation: $y' - 4x\, y = 4x\, y^n$, with $n = 1/2$.

It is separable: $\dfrac{y'}{y + \sqrt{y}} = 4x$.

The equation is not homogeneous. It is not exact.

Although the equation is both separable and Bernoulli, it is not simple to integrate using the separable equation method. Indeed

$$\int \frac{y'}{y + \sqrt{y}}\, dt = \int 4x\, dx + c \quad \Rightarrow \quad \int \frac{dy}{y + \sqrt{y}} = 2x^2 + c.$$

The integral on the left-hand side requires an integration table.

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Example

Find every solution of the initial value problem

$$y' = 4x(y + \sqrt{y}), \qquad y(0) = 4.$$

Solution: We find solutions using the Bernoulli method.

$$y' - 4x\, y = 4x\, y^{1/2} \quad \Rightarrow \quad \frac{y'}{y^{1/2}} - 4x\, y^{1/2} = 4x.$$

Change the unknowns: $v = 1/y^{n-1}$, with $n = 1/2$. That is,

$$v = \frac{1}{y^{-1/2}} \quad \Rightarrow \quad v = y^{1/2}, \quad \Rightarrow \quad v' = \frac{1}{2}\,\frac{y'}{y^{1/2}}.$$

$$2v' - 4xv = 4x \quad \Rightarrow \quad v' - 2xv = 2x.$$

The coefficient function is $a(x) = -2x$, so $A(x) = -x^2$, and the integrating factor is $\mu(x) = e^{-x^2}$.

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Example

Find every solution of the initial value problem

$$y' = 4x(y + \sqrt{y}), \qquad y(0) = 4.$$

Solution: Recall: $v' - 2xv = 2x$ and $\mu(x) = e^{-x^2}$.

$$e^{-x^2} v' - 2xe^{-x^2} v = 2x\, e^{-x^2} \overset{\text{Verify!}}{\Longrightarrow} \big(e^{-x^2} v\big)' = 2xe^{-x^2}.$$

$$e^{-x^2} v = \int 2xe^{-x^2}\, dx + c \quad \Rightarrow \quad e^{-x^2} v = -e^{-x^2} + c.$$

We conclude that $v = c\, e^{x^2} - 1$. The initial condition for y implies the initial condition for v, that is, $v(x) = \sqrt{y(x)}$ implies $v(0) = 2$.

$$2 = v(0) = c - 1 \quad \Rightarrow \quad c = 3 \quad \Rightarrow \quad v(x) = 3e^{x^2} - 1.$$

We finally find $y = v^2$, that is, $y(x) = (3e^{x^2} - 1)^2$. $\triangleleft$

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Example

Find the domain of the function y solution of the IVP

$$y' = -\frac{2t}{y}, \qquad y(1) = 2.$$

Solution: We first need to find the solution y.
The equation is separable.

$$y\,y' = -2t \quad \Rightarrow \quad \int y\,y'\,dt = \int -2t\,dt + c \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + c$$

$$\frac{4}{2} = \frac{y^2(1)}{2} = -1 + c \quad \Rightarrow \quad c = 3 \quad \Rightarrow \quad y(t) = \sqrt{2(3 - t^2)}.$$

The domain of the solution y is $D = (-\sqrt{3}, \sqrt{3})$.

The points $\pm\sqrt{3}$ do not belong to the domain of y, since y' and the differential equation are not defined there. ◁

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Example

Find the domain of the function y solution of the IVP

$$y' = -\frac{2t}{y}, \qquad y(t_0) = y_0 > 0.$$

Solution: The solution y is given as above, $\dfrac{y^2}{2} = -t^2 + c$.
The initial condition implies

$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \;\Rightarrow\; c = \frac{y_0^2}{2} + t_0^2 \;\Rightarrow\; \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}.$$

The solution to the IVP is $y(t) = \sqrt{2(t_0^2 - t^2) + y_0^2}$.

The domain of the solution depends on the initial condition t_0, y_0:

$$D = \Big(-\sqrt{t_0^2 + \frac{y_0^2}{2}}, +\sqrt{t_0^2 + \frac{y_0^2}{2}}\Big).$$

◁

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Example

Find every solution y to the equation $y' = -\dfrac{2x + 3y}{3x + 4y}$.

Solution: The equation is not linear, not Bernoulli, not separable. It is homogeneous. (Multiply numerator and denominator on the right hand side by $(1/x)$.)
Is it exact? $(3x + 4y)\, y' + (2x + 3y) = 0$ implies $\partial_x N = 3 = \partial_y M$. So the equation is exact.

We choose here the exact equation method. (Finding the potential function is sometimes simpler that solving homogeneous Eqs.)

We need to find the potential function ψ:

$$\partial_y \psi = N \quad \Rightarrow \quad \psi = 3xy + 2y^2 + g(x).$$

$$\partial_x \psi = M \quad \Rightarrow \quad 3y + g'(x) = 2x + 3y \quad \Rightarrow \quad g(x) = x^2.$$

We conclude: $\psi(x, y) = 3xy + 2y^2 + x^2$, and $\psi(x, y(x)) = c$. $\triangleleft$

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Example

Find every solution y to the equation $y' = -\dfrac{2x + 3y}{3x + 4y}$.

Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation. We just start the calculation to see the difficulty:

$$y' = -\frac{(2x + 3y)}{(3x + 4y)} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = -\frac{2 + 3\left(\frac{y}{x}\right)}{3 + 4\left(\frac{y}{x}\right)}.$$

The change $v = y/x$ implies $y = xv$ and $y' = v + x\, v'$. Hence

$$v + x\, v' = \frac{2 + 3v}{3 + 4v} \quad \Rightarrow \quad x\, v' = \frac{2 + 3v}{3 + 4v} - v = \frac{2 + 3v - 3v + 4v^2}{3 + 4v}.$$

We conclude that v satisfies $\dfrac{3 + 4v}{2 - 4v^2}\, v' = \dfrac{1}{x}$.

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Example

Find every solution y to the equation $y' = -\dfrac{2x + 3y}{3x + 4y}$.

Solution: Recall: $\dfrac{3 + 4v}{2 - 4v^2}\, v' = \dfrac{1}{x}$.

This equation is complicated to integrate.

$$\int \frac{3\, v'}{2 - 4v^2}\, dx + \int \frac{4v\, v'}{2 - 4v^2}\, dx = \int \frac{1}{x}\, dx + c = \ln(x) + c.$$

The usual substitution $u = v(x)$ implies $du = v'\, dx$, so

$$\int \frac{3\, du}{2 - 4u^2} + \int \frac{4u\, du}{2 - 4u^2} = \ln(x) + c.$$

The first integral on the left-hand side requires integration tables.

This is why the exact method is simpler to use in this case. $\triangleleft$

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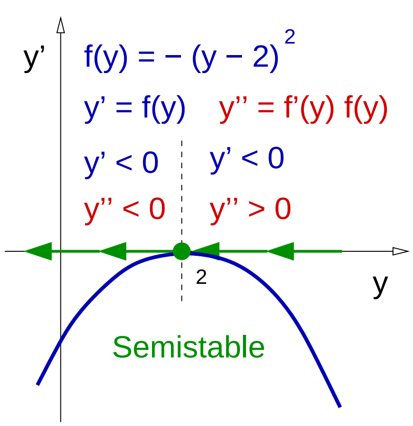
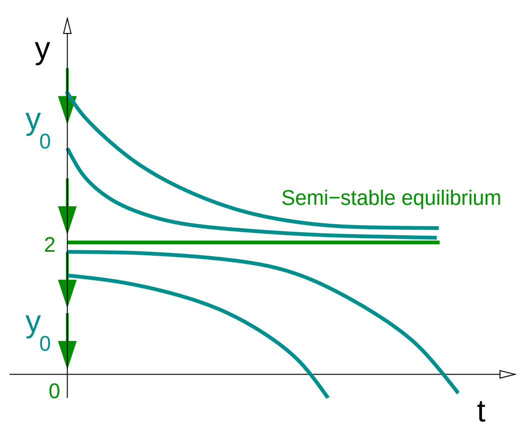
Example

Sketch the graph of the function y solution of $y' = -(y - 2)^2$ for initial data $y(0) = y_0 \in \mathbb{R}$.

Solution: We first plot the function $f(y) = -(y - 2)^2$.
Find the equilibrium solutions, $f(y) = 0$.
Determine the increasing-decreasing intervals for y. (Sign of y'.)
Determine the curvature of y. (Sign of y''.)





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Example

Assume that $r_i = r_o = r$ and q_i are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find t_1 such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Since $r_i = r_o$, we get $V'(t) = 0$, so $V(t) = V_0$.

The equation for Q is $Q' = rq_i - (r/V_0)\, Q$.

Since $q_i = 0$, $Q' = -(r/V_0)\, Q$. The solution is $Q(t) = Q_0\, e^{-rt/V_0}$.

We now look for t_1 solution of

$$\frac{Q_0}{V_0}\frac{1}{100} = q(t_1) = \frac{Q_0}{V_0}\, e^{-rt_1/V_0} \quad \Rightarrow \quad e^{-rt_1/V_0} = \frac{1}{100}.$$

$$-\frac{r}{V_0}\, t_1 = \ln\Bigl(\frac{1}{100}\Bigr) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0}\, t_1 = \ln(100).$$

We conclude that $t_1 = \dfrac{V_0}{r}\ln(100)$. Hence: $t_1 = 100\ln(100)$. $\triangleleft$