Variable coefficients second order linear ODE (Sect. 3.2).

Summary: The study the main properties of solutions to second order, linear, variable coefficients, ODE.

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel's theorem on the Wronskian.

Review: Second order linear ODE.

Definition

Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t)$$

is called a *second order linear* differential equation with *variable coefficients*.

Theorem

If the functions y_1 and y_2 are solutions to the homogeneous linear equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0,$$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants c_1, $c_2 \in \mathbb{R}$.

Variable coefficients second order linear ODE (Sect. 3.2).

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Existence and uniqueness of solutions.

Theorem (Variable coefficients)

If the functions $a, b : (t_1, t_2) \to \mathbb{R}$ are continuous, the constants $t_0 \in (t_1, t_2)$ and $y_0, y_1 \in \mathbb{R}$, then there exists a unique solution $y : (t_1, t_2) \to \mathbb{R}$ to the initial value problem

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

Remarks:

- Unlike the first order linear ODE where we have an explicit expression for the solution, there is no explicit expression for the solution of second order linear ODE.
- Two integrations must be done to find solutions to second order linear. Therefore, initial value problems with two initial conditions can have a unique solution.

Existence and uniqueness of solutions.

Example

Find the longest interval $I \in \mathbb{R}$ such that there exists a unique solution to the initial value problem

$$(t-1)y'' - 3ty' + 4y = t(t-1), \qquad y(-2) = 2, \qquad y'(-2) = 1.$$

Solution: We first write the equation above in the form given in the Theorem above,

$$y'' - \frac{3t}{t-1}\,y' + \frac{4}{t-1}\,y = t.$$

The intervals where the hypotheses in the Theorem above are satisfied, that is, where the equation coefficients are continuous, are $I_1 = (-\infty, 1)$ and $I_2 = (1, \infty)$. Since the initial condition belongs to I_1, the solution domain is

$$I_1 = (-\infty, 1). \qquad \triangleleft$$

Existence and uniqueness of solutions.

Remarks:

- Every solution of the first order linear equation

$$y' + a(t)\,y = 0$$

is given by $y(t) = c\,e^{-A(t)}$, with $A(t) = \int a(s)\,ds$.

- All solutions above are proportional to each other:

$$y_1(t) = c_1\,e^{-A(t)}, \quad y_2(t) = c_2\,e^{-A(t)} \;\Rightarrow\; y_1(t) = \frac{c_1}{c_2}\,y_2(t)$$

Remark: The above statement is *not true* for solutions of second order, linear, homogeneous equations, $y'' + a_1(t)\,y' + a_0(t)y = 0$. Before we prove this statement we need few definitions:

- Proportional functions (linearly dependent).
- Wronskian of two functions.

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Linearly dependent and independent functions.

Definition

Two continuous functions $y_1, y_2 : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ are called *linearly dependent, (ld),* on the interval (t_1, t_2) iff there exists a constant c such that for all $t \in I$ holds

$$y_1(t) = c\, y_2(t).$$

The two functions are called *linearly independent, (li),* on the interval (t_1, t_2) iff they are not linearly dependent.

Remarks:

- $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ are ld $\Leftrightarrow$ there exist constants c_1, c_2, not both zero, such that $c_1\, y_1(t) + c_2\, y_2(t) = 0$ for all $t \in (t_1, t_2)$.
- $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ are li $\Leftrightarrow$ the only constants c_1, c_2, solutions of $c_1\, y_1(t) + c_2\, y_2(t) = 0$ for all $t \in (t_1, t_2)$ are $c_1 = c_2 = 0$.
- These definitions are not given in the textbook.

Linearly dependent and independent functions.

Example

(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are ld.

(b) Show that $y_1(t) = \sin(t)$, $y_2(t) = t\sin(t)$ are li.

Solution:

Case (a): Trivial. $y_2 = 2y_1$.

Case (b): Find constants c_1, c_2 such that for all $t \in \mathbb{R}$ holds

$$c_1 \sin(t) + c_2 t \sin(t) = 0 \quad \Leftrightarrow \quad (c_1 + c_2 t) \sin(t) = 0.$$

Evaluating at $t = \pi/2$ and $t = 3\pi/2$ we obtain

$$c_1 + \frac{\pi}{2} c_2 = 0, \quad c_1 + \frac{3\pi}{2} c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.$$

We conclude: The functions y_1 and y_2 are li. $\triangleleft$

Variable coefficients second order linear ODE (Sect. 3.2).

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The Wronskian of two functions.

Remark: The Wronskian is a function that determines whether two functions are ld or li.

Definition

The *Wronskian* of functions $y_1,\ y_2 : (t_1, t_2) \to \mathbb{R}$ is the function

$$W_{y_1 y_2}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Remark:

- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, then $W_{y_1 y_2}(t) = \det\big(A(t)\big)$.
- An alternative notation is: $W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

The Wronskian of two functions.

Example

Find the Wronskian of the functions:

(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$. (ld)

(b) $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t)$. (li)

Solution:

Case (a): $W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix}$. Therefore,

$$W_{y_1 y_2}(t) = \sin(t)2\cos(t) - \cos(t)2\sin(t) \quad \Rightarrow \quad W_{y_1 y_2}(t) = 0.$$

Case (b): $W_{y_1 y_2} = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix}$. Therefore,

$$W_{y_1 y_2}(t) = \sin(t)\big[\sin(t) + t\cos(t)\big] - \cos(t)t\sin(t).$$

We obtain $W_{y_1 y_2}(t) = \sin^2(t)$. ◁

The Wronskian of two functions.

Remark: The Wronskian determines whether two functions are linearly dependent or independent.

Theorem (Wronskian and linearly dependence)

The continuously differentiable functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ are linearly dependent iff $W_{y_1 y_2}(t) = 0$ for all $t \in (t_1, t_2)$.

Remark: Importance of the Wronskian:

- Sometimes it is not simple to decide whether two functions are proportional to each other.
- The Wronskian is useful to study properties of solutions to ODE without having the explicit expressions of these solutions. (See Abel's Theorem later on.)

The Wronskian of two functions.

Example

Show whether the following two functions form a l.d. or l.i. set:

$$y_1(t) = \cos(2t) - 2\cos^2(t), \qquad y_2(t) = \cos(2t) + 2\sin^2(t).$$

Solution: Compute their Wronskian:

$$W_{y_1 y_2}(t) = y_1\, y_2' - y_1'\, y_2.$$

$$W_{y_1 y_2}(t) = \big[\cos(2t) - 2\cos^2(t)\big]\,\big[-2\sin(2t) + 4\sin(t)\cos(t)\big] - \big[-2\sin(2t) + 4\sin(t)\cos(t)\big]\,\big[\cos(2t) + 2\sin^2(t)\big].$$

$$\sin(2t) = 2\sin(t)\cos(t) \Rightarrow \big[-2\sin(2t) + 4\sin(t)\cos(t)\big] = 0.$$

We conclude $W_{y_1 y_2}(t) = 0$, so the functions y_1 and y_2 are ld. $\triangleleft$

The Wronskian of two functions.

Theorem (Variable coefficients)

- *If a_1, a_0, $b : (t_1, t_2) \to \mathbb{R}$ are continuous, then there exist two linearly independent solutions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ to the equation*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t). \tag{1}$$

- *Every other solution y of Eq. (1) can be decomposed as*

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

for appropriate constants c_1, c_2.

- *For every constant $t_0 \in (t_1, t_2)$ and y_0, $y_1 \in \mathbb{R}$, there exists a unique solution $y : (t_1, t_2) \to \mathbb{R}$ to the initial value problem given by Eq. (1) with the initial conditions*

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

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General and fundamental solutions.

Remark: The Theorem above justifies the following definitions.

Definition

Two solutions y_1, y_2 of the homogeneous equation

$$y'' + a_1(t)y' + a_0(t)y = 0,$$

are called *fundamental solutions* iff the functions y_1, y_2 are linearly independent, that is, iff $W_{y_1 y_2} \neq 0$.

Definition

Given any two fundamental solutions y_1, y_2, and arbitrary constants c_1, c_2, the function

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

is called the *general solution* of Eq. (1).

General and fundamental solutions.

Example

Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2\, y'' + 3t\, y' - y = 0.$$

Solution: First show that y_1 is a solution:

$$y_1 = t^{1/2}, \quad y_1' = \frac{1}{2}\, t^{-1/2}, \quad y_1'' = -\frac{1}{4}\, t^{-3/2},$$

$$2t^2\left(-\frac{1}{4}\, t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2} t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}} = -\frac{1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0.$$

Now show that y_2 is a solution:

$$y_2 = t^{-1}, \quad y_2' = -t^{-2}, \quad y_2'' = 2t^{-3},$$

$$2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0.$$

General and fundamental solutions.

Example

Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2\, y'' + 3t\, y' - y = 0.$$

Solution: We show that y_1, y_2 are linearly independent.

$$W_{y_1 y_2}(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}\, t^{-1/2} & -t^{-2} \end{vmatrix}.$$

$$W_{y_1 y_2}(t) = -t^{1/2}\, t^{-2} - \frac{1}{2} t^{-1/2}\, t^{-1} = -t^{-3/2} - \frac{1}{2} t^{-3/2}$$

$$W_{y_1 y_2}(t) = -\frac{3}{3}\, t^{-3/2} \quad \Rightarrow \quad y_1,\ y_2 \text{ li.} \qquad \triangleleft$$

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Abel's theorem on the Wronskian.

Theorem (Abel)

If a_1, $a_0 : (t_1, t_2) \to \mathbb{R}$ are continuous functions and y_1, y_2 are continuously differentiable solutions of the equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0,$$

then the Wronskian $W_{y_1 y_2}$ is a solution of the equation

$$W'_{y_1 y_2}(t) + a_1(t)\, W_{y_1 y_2}(t) = 0.$$

Therefore, for any $t_0 \in (t_1, t_2)$, the Wronskian $W_{y_1 y_2}$ is given by

$$W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, e^{A(t)} \qquad A(t) = \int_{t_0}^{t} a_1(s)\, ds.$$

Remarks: If the the Wronskian of two solutions vanishes at the initial time, then it vanishes at all times.

Abel's theorem on the Wronskian.

Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: Write the equation as in Abel's Theorem,

$$y'' - \left(\frac{2}{t} + 1\right) y' + \left(\frac{2}{t^2} + \frac{1}{t}\right) y = 0.$$

Abel's Theorem says that the Wronskian satisfies the equation

$$W'_{y_1 y_2}(t) - \left(\frac{2}{t} + 1\right) W_{y_1 y_2}(t) = 0.$$

This is a first order, linear equation for $W_{y_1 y_2}$. The integrating factor method implies

$$A(t) = -\int_{t_0}^{t} \left(\frac{2}{s} + 1\right) ds = -2\ln\left(\frac{t}{t_0}\right) - (t - t_0)$$

Abel's theorem on the Wronskian.

Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: $A(t) = -2\ln\Big(\frac{t}{t_0}\Big) - (t - t_0) = \ln\Big(\frac{t_0^2}{t^2}\Big) - (t - t_0).$

The integrating factor is $\mu = \frac{t_0^2}{t^2}\, e^{-(t-t_0)}$. Therefore,

$$\Big[\mu(t) W_{y_1 y_2}(t)\Big]' = 0 \quad \Rightarrow \quad \mu(t) W_{y_1 y_2}(t) - \mu(t_0) W_{y_1 y_2}(t_0) = 0$$

so, the solution is $W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, \frac{t^2}{t_0^2}\, e^{(t-t_0)}$.

Denoting $c = \big(W_{y_1 y_2}(t_0)/t_0^2\big)\, e^{-t_0}$, then $W_{y_1 y_2}(t) = c\, t^2 e^t$. $\lhd$