Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
- Superposition property.
- Constant coefficients equations.
- The characteristic equation.
- The main result.

Second order linear differential equations.

Definition

Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t) \qquad (1)$$

is called a *second order linear* differential equation with *variable coefficients*. The equation in (1) is called *homogeneous* iff for all $t \in \mathbb{R}$ holds

$$b(t) = 0.$$

The equation in (1) is called of *constant coefficients* iff a_1, a_0, and b are constants.

Remark: The notion of an homogeneous equation presented here is not the same as the notion presented in the previous chapter.

Second order linear differential equations.

Example

(a) A second order, linear, homogeneous, constant coefficients equation is

$$y'' + 5y' + 6 = 0.$$

(b) A second order order, linear, constant coefficients, non-homogeneous equation is

$$y'' - 3y' + y = 1.$$

(c) A second order, linear, non-homogeneous, variable coefficients equation is

$$y'' + 2t\, y' - \ln(t)\, y = e^{3t}.$$

(d) Newton's second law of motion ($ma = f$) for point particles of mass m moving in one space dimension under a force $f : \mathbb{R} \to \mathbb{R}$ is given by

$$m\, y''(t) = f(t).$$

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Superposition property.

Theorem

If the functions y_1 and y_2 are solutions to the homogeneous linear equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \qquad (2)$$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants $c_1, c_2 \in \mathbb{R}$.

Proof: Verify that the function $y = c_1 y_1 + c_2 y_2$ satisfies Eq. (2) for every constants c_1, c_2, that is,

$$(c_1 y_1 + c_2 y_2)'' + a_1(t)(c_1 y_1 + c_2 y_2)' + a_0(t)(c_1 y_1 + c_2 y_2)$$

$$= (c_1 y_1'' + c_2 y_2'') + a_1(t)(c_1 y_1' + c_2 y_2') + a_0(t)(c_1 y_1 + c_2 y_2)$$

$$= c_1 \big[y_1'' + a_1(t) y_1' + a_0(t) y_1 \big] + c_2 \big[y_2'' + a_1(t) y_2' + a_0(t) y_2 \big] = 0.$$

□

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Constant coefficients equations.

Remark: Just by trial and error one can find solutions to second order, constant coefficients, homogeneous, linear differential equations. We present the main ideas with an example.

Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: We look for solutions proportional to exponentials e^{rt}, for an appropriate constant $r \in R$, since the exponential can be canceled out from the equation.
If $y(t) = e^{rt}$, then $y'(t) = re^{rt}$, and $y''(t) = r^2 e^{rt}$. Hence

$$(r^2 + 5r + 6)e^{rt} = 0 \quad \Leftrightarrow \quad r^2 + 5r + 6 = 0.$$

That is, r must be a root of the polynomial $p(r) = r^2 + 5r + 6$.

This polynomial is called the characteristic polynomial of the differential equation.

Constant coefficients equations.

Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

The roots of the characteristic polynomial are

$$r = \frac{1}{2}\left(-5 \pm \sqrt{25 - 24}\right) = \frac{1}{2}(-5 \pm 1) \quad \Rightarrow \quad \begin{cases} r_1 = -2, \\ r_2 = -3. \end{cases}$$

Therefore, we have found two solutions to the ODE,

$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{-3t}.$$

Their superposition provides infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}. \qquad \triangleleft$$

Constant coefficients equations.

Summary: The differential equation $y'' + 5y' + 6y = 0$ has infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Remarks:

- There are two free constants in the solution found above.
- The ODE above is second order, so two integrations must be done to find the solution. This explain the origin of the two free constant in the solution.
- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.

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The characteristic equation.

Definition

Given a second order linear homogeneous differential equation with constant coefficients

$$y'' + a_1 y' + a_0 = 0, \tag{3}$$

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (3) are, respectively,

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

If r_1, r_2 are the solutions of the characteristic equation and c_1, c_2 are constants, then the function

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is called the *general solution* of the Eq. (3).

The characteristic equation.

Example

Find the solution y of the initial value problem

$$y'' + 5y' + 6 = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

Solution: A solution of the differential equation above is

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

We now find the constants c_1 and c_2 that satisfy the initial conditions above:

$$1 = y(0) = c_1 + c_2, \qquad -1 = y'(0) = -2c_1 - 3c_2.$$

$$c_1 = 1 - c_2 \Rightarrow 1 = 2(1 - c_2) + 3c_2 \Rightarrow c_2 = -1 \Rightarrow c_1 = 2.$$

Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

$\triangleleft$

The characteristic equation.

Example

Find the general solution y of the differential equation

$$2y'' - 3y' + y = 0.$$

Solution: We look for every solution of the form $y(t) = e^{rt}$, where r is a solution of the characteristic equation

$$2r^2 - 3r + 1 = 0 \;\Rightarrow\; r = \frac{1}{4}\big(3 \pm \sqrt{9-8}\big) \;\Rightarrow\; \begin{cases} r_1 = 1, \\ r_2 = \dfrac{1}{2}. \end{cases}$$

Therefore, the general solution of the equation above is

$$y(t) = c_1 e^t + c_2 e^{t/2},$$

where c_1, c_2 are arbitrary constants. ◁

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The main result.

Theorem (Constant coefficients)

Given real constants a_1, a_0, *consider the homogeneous, linear differential equation on the unknown* $y : \mathbb{R} \to \mathbb{R}$ *given by*

$$y'' + a_1\, y' + a_0\, y = 0. \tag{4}$$

Let r_+, r_- *be the roots of the characteristic polynomial* $p(r) = r^2 + a_1 r + a_0$, *and let* c_0, c_1 *be arbitrary constants. Then, any solution of Eq. (4) belongs to only one of the following cases:*

(a) *If* $r_+ \neq r_-$, *the general solution is* $y(t) = c_0 e^{r_+ t} + c_1 e^{r_- t}$.

(b) *If* $r_+ = r_- \in \mathbb{R}$, *the general solution is* $y(t) = (c_0 + c_1 t) e^{r_+ t}$.

Furthermore, given real constants t_0, y_0 *and* y_1, *there is a unique solution to the initial value problem given by Eq. (4) and the initial conditions*

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$