On linear and non-linear equations.(Sect. 2.4).

- Review: Linear differential equations.
- Non-linear differential equations.
- Properties of solutions to non-linear ODE.
- The Bernoulli equation.

Review: Linear differential equations.

Theorem (Variable coefficients)

Given continuous functions $a, b : (t_1, t_2) \to \mathbb{R}$, with $t_2 > t_1$, and given constants $t_0 \in (t_1, t_2)$, $y_0 \in \mathbb{R}$, the IVP

$$y' = -a(t)\, y + b(t), \qquad y(t_0) = y_0,$$

has the unique solution $y : (t_1, t_2) \to \mathbb{R}$ given by

$$y(t) = \frac{1}{\mu(t)} \Big[y_0 + \int_{t_0}^{t} \mu(s)\, b(s)\, ds \Big], \tag{1}$$

where the integrating factor function is given by

$$\mu(t) = e^{A(t)}, \qquad A(t) = \int_{t_0}^{t} a(s)\, ds.$$

Proof: Based on the integration factor method.

Review: Linear differential equations.

Remarks:

- The Theorem above assumes that the coefficients a, b, are continuous in $(t_1, t_2) \subset \mathbb{R}$.
- The Theorem above implies:
  - (a) There is an explicit expression for the solutions of a linear IVP, given in Eq. (1).
  - (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  - (c) For every initial condition $y_0 \in \mathbb{R}$ the corresponding solution $y(t)$ of a linear IVP is defined for all $t \in (t_1, t_2)$.
- None of these properties holds for solutions to non-linear differential equations.

On linear and non-linear equations.(Sect. 2.4).

- Review: Linear differential equations.
- **Non-linear differential equations.**
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Non-linear differential equations.

Definition

An ordinary differential equation $y'(t) = f(t, y(t))$ is called *non-linear* iff the function $(t, u) \mapsto f(t, u)$ is non-linear in the second argument.

Example

(a) The differential equation $y'(t) = \dfrac{t^2}{y^3(t)}$ is non-linear, since the function $f(t, u) = t^2/u^3$ is non-linear in the second argument.

(b) The differential equation $y'(t) = 2ty(t) + \ln\big(y(t)\big)$ is non-linear, since the function $f(t, u) = 2tu + \ln(u)$ is non-linear in the second argument, due to the term $\ln(u)$.

(c) The differential equation $\dfrac{y'(t)}{y(t)} = 2t^2$ is linear, since the function $f(t, u) = 2t^2 u$ is linear in the second argument.

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Properties of solutions to non-linear ODE.

Theorem (Non-linear ODE)

Fix a non-empty rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ and fix a function $f : R \to \mathbb{R}$ denoted as $(t, u) \mapsto f(t, u)$. If the functions f and $\partial_u f$ are continuous on R, and $(t_0, y_0) \in R$, then there exists a smaller open rectangle $\hat{R} \subset R$ with $(t_0, y_0) \in \hat{R}$ such that the IVP

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0$$

has a unique solution y on the set $\hat{R} \subset \mathbb{R}^2$.

Remarks:

(i) There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.

(ii) Non-uniqueness of solution to the IVP above may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.

(iii) Changing the initial data y_0 may change the domain on the variable t where the solution $y(t)$ is defined.

Properties of solutions to non-linear ODE.

Example

Given non-zero constants a_1, a_2, a_3, a_4, find every solution y of

$$y' = \frac{t^2}{\left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right)}.$$

Solution: The ODE is separable. So first, rewrite the equation as

$$\left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right) y' = t^2,$$

then we integrate in t on both sides of the equation,

$$\int \left(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\right) y'\, dt = \int t^2\, dt + c.$$

Introduce the substitution $u = y(t)$, so $du = y'(t)\, dt$,

$$\int \left(u^4 + a_4\, u^3 + a_3\, u^2 + a_2\, u + a_1\right) du = \int t^2\, dt + c.$$

Properties of solutions to non-linear ODE.

Example

Given non-zero constants a_1, a_2, a_3, a_4, find every solution y of

$$y' = \frac{t^2}{\big(y^4 + a_4\, y^3 + a_3\, y^2 + a_2\, y + a_1\big)}.$$

Solution:

Recall: $\displaystyle\int \big(u^4 + a_4\, u^3 + a_3\, u^2 + a_2\, u + a_1\big)\, du = \int t^2\, dt + c.$

Integrate, and in the result substitute back the function y:

$$\frac{1}{5}\, y^5(t) + \frac{a_4}{4}\, y^4(t) + \frac{a_3}{3}\, y^3(t) + \frac{a_2}{2}\, y^2(t) + a_1\, y(t) = \frac{t^3}{3} + c.$$

The solution is in implicit form. It is the root of a polynomial degree five. There is no formula for the roots of a general polynomial degree five or bigger.

There is no explicit expression for solutions y of the ODE. $\triangleleft$

Properties of solutions to non-linear ODE.

Example

Find every solution y of the initial value problem

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

Remark: The equation above is non-linear, separable, and the function $f(t,u) = u^{1/3}$ has derivative

$$\partial_u f = \frac{1}{3}\, \frac{1}{u^{2/3}},$$

so $\partial_u f$ is not continuous at $u = 0$.

The initial condition above is precisely where f is not continuous.

Solution: There are two solutions to the IVP above:

The first solution is

$$y_1(t) = 0.$$

Properties of solutions to non-linear ODE.

Example

Find every solution y of the initial value problem

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

Solution: The second solution is obtained as follows:

$$\int \big[y(t)\big]^{-1/3}\, y'(t)\, dt = \int dt + c.$$

Then, the substitution $u = y(t)$, with $du = y'(t)\, dt$, implies that

$$\int u^{-1/3}\, du = \int dt + c \quad \Rightarrow \quad \frac{3}{2}\big[y(t)\big]^{2/3} = t + c,$$

$$y(t) = \Big[\frac{2}{3}(t + c)\Big]^{3/2} \Rightarrow 0 = y(0) = \Big(\frac{2}{3}c\Big)^{3/2} \Rightarrow c = 0.$$

So, the second solution is: $y_2(t) = \Big(\frac{2}{3}t\Big)^{3/2}$. Recall $y_1(t) = 0$. $\triangleleft$

Properties of solutions to non-linear ODE.

Example

Find the solution y to the initial value problem

$$y'(t) = y^2(t), \qquad y(0) = y_0.$$

Solution: This is a separable equation. So,

$$\int \frac{y'\, dt}{y^2} = \int dt + c \quad \Rightarrow \quad -\frac{1}{y} = t + c \quad \Rightarrow \quad y(t) = -\frac{1}{t + c}.$$

Using the initial condition in the expression above,

$$y_0 = y(0) = -\frac{1}{c} \quad \Rightarrow \quad c = -\frac{1}{y_0} \quad \Rightarrow \quad y(t) = \frac{1}{\Big(\frac{1}{y_0} - t\Big)}.$$

This solution diverges at $t = 1/y_0$, so its domain is $\mathbb{R} - \{y_0\}$.

The solution domain depends on the values of the initial data y_0. $\triangleleft$

Properties of solutions to non-linear ODE.

Summary:

- Linear ODE:
  (a) There is an explicit expression for the solution of a linear IVP.
  (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  (c) The domain of the solution of a linear IVP is defined for every initial condition $y_0 \in \mathbb{R}$.

- Non-linear ODE:
  (i) There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.
  (ii) Non-uniqueness of solution to a non-linear IVP may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.
  (iii) Changing the initial data y_0 may change the domain on the variable t where the solution $y(t)$ is defined.

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The Bernoulli equation.

Remark: The Bernoulli equation is a non-linear differential equation that can be transformed into a linear differential equation.

Definition

Given functions $p, q : \mathbb{R} \to \mathbb{R}$ and a real number n, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y' + p(t)\, y = q(t)\, y^n$$

is called the *Bernoulli equation*.

Theorem

The function $y : \mathbb{R} \to \mathbb{R}$ is a solution of the Bernoulli equation for

$$y' + p(t)\, y = q(t)\, y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' - (n-1)p(t)\, v = -(n-1)q(t).$$

The Bernoulli equation.

Example

Given arbitrary constants a_0 and b_0, find every solution of the differential equation

$$y' = a_0 y + b_0 y^3.$$

Solution: This is a Bernoulli equation. Divide the equation by y^3,

$$\frac{y'}{y^3} = \frac{a_0}{y^2} + b_0.$$

Introduce the function $v = 1/y^2$, with derivative $v' = -2(y'/y^3)$, into the differential equation above,

$$-\frac{v'}{2} = a_0 v + b_0 \quad \Rightarrow \quad v' = -2a_0 v - 2b_0 \quad \Rightarrow \quad v' + 2a_0 v = -2b_0.$$

The Bernoulli equation.

Example

Given arbitrary constants a_0 and b_0, find every solution of the differential equation

$$y' = a_0 y + b_0 y^3.$$

Solution: Recall: $v' + 2a_0 v = -2b_0$.

The last equation is a linear differential equation for v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(t) = e^{2a_0 t}$,

$$\big(e^{2a_0 t} v\big)' = -2b_0\, e^{2a_0 t} \quad \Rightarrow \quad e^{2a_0 t} v = -\frac{b_0}{a_0}\, e^{2a_0 t} + c$$

We obtain that $v = c\, e^{-2a_0 t} - \dfrac{b_0}{a_0}$. Since $v = 1/y^2$,

$$\frac{1}{y^2} = c\, e^{-2a_0 t} - \frac{b_0}{a_0} \quad \Rightarrow \quad y = \pm \frac{1}{\big(c\, e^{-2a_0 t} - \frac{b_0}{a_0}\big)^{1/2}}. \qquad \triangleleft$$