Autonomous systems (Sect. 2.5).

- Definition and examples.
- Qualitative analysis of the solutions.
- Equilibrium solutions and stability.
- Population growth equation.

Definition and examples

Definition

A first order ODE on the unknown function $y : \mathbb{R} \to \mathbb{R}$ is called *autonomous* iff the ODE has the form

$$\frac{dy}{dt} = f(y).$$

Remark:

- The independent variable, t, does not appear explicitly in an autonomous ODE.
- Autonomous systems are a particular case of separable equations,

$$h(y)\,y' = g(t), \qquad g(t) = 1, \quad f(y) = \frac{1}{h(y)}.$$

- It is simple to study the qualitative properties of solutions to autonomous systems.

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Qualitative analysis of the solutions

Remark: It is simple to study the qualitative properties of solutions to autonomous systems.

Example

Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

Solution: One way: Find the exact solutions and then graph them.

The equation is separable, then

$$\frac{y'(t)}{\sin\big[y(t)\big]} = 1 \quad \Rightarrow \quad \int_0^t \frac{y'(t)}{\sin\big[y(t)\big]} = t$$

Use the substitution $u = y(t)$, $du = y'(t)\,dt$,

$$\int_{y_0}^{y(t)} \frac{du}{\sin(u)} = t \quad \Rightarrow \quad \ln\Big[\frac{\sin(u)}{1+\cos(u)}\Big]\Big|_{y_0}^{y(t)} = t.$$

Qualitative analysis of the solutions

Example

Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

Solution: Recall: $\ln\left[\frac{\sin(u)}{1+\cos(u)}\right]\Big|_{y_0}^{y(t)} = t.$

$$\ln\left[\frac{\sin(y)}{1+\cos(y)}\right] - \ln\left[\frac{\sin(y_0)}{1+\cos(y_0)}\right] = t.$$

$$\ln\left[\frac{\sin(y)}{[1+\cos(y)]}\,\frac{[1+\cos(y_0)]}{\sin(y_0)}\right] = t.$$

The implicit expression of the solution is

$$\frac{\sin(y)}{[1+\cos(y)]} = \frac{\sin(y_0)}{[1+\cos(y_0)]}\,e^t.$$

Without a computer it is difficult to graph the solution.

Qualitative analysis of the solutions

Example

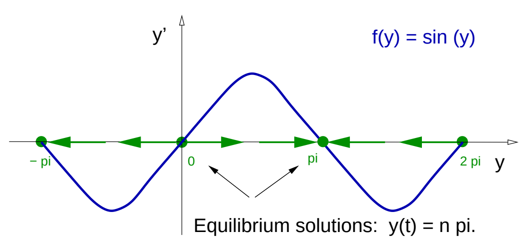
Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

Solution: Recall: $\frac{\sin(y)}{[1+\cos(y)]} = \frac{\sin(y_0)}{[1+\cos(y_0)]}\,e^t.$

Another way:

(1) Plot the function

$$f(y) = \sin(y).$$



(2) Find the zeros of f. Since $f(y) = \sin(y) = 0$, then $y = m\pi$.

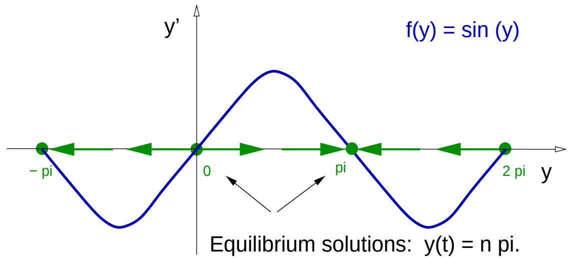
The constants $y = m\pi$, are solutions of $y' = \sin(y)$.
They are called equilibrium solutions.

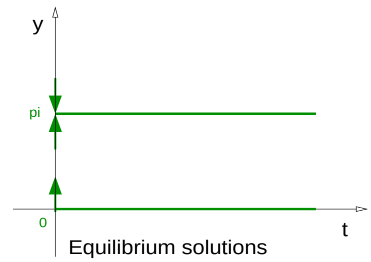
Qualitative analysis of the solutions

Example

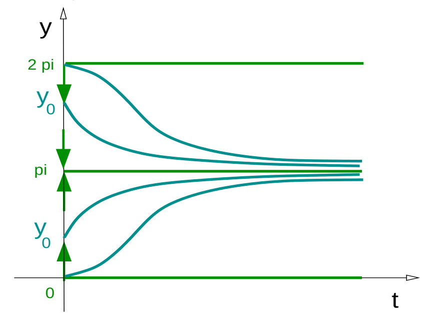
Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

Solution:





(3) The solution is:
Increasing for $y' = \sin(y) > 0$,
Decreasing for $y' = \sin(y) < 0$.



Autonomous systems (Sect. 2.5).

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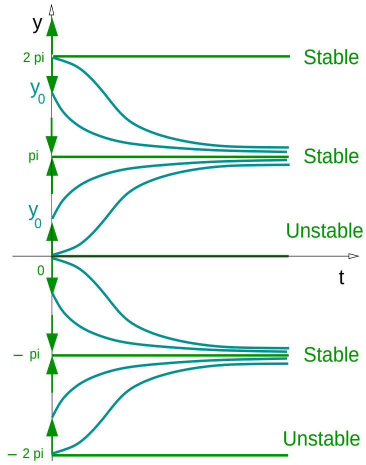
- Qualitative analysis of the solutions.
- **Equilibrium solutions and stability.**
- Population growth equation.

Equilibrium solutions and stability



Definition

The constant y_0 is an *equilibrium solution* of the autonomous system $y' = f(y)$ iff hold that $f(y_0) = 0$.

The equilibrium solution y_0 is *asymptotically stable* iff there exists $I = (y_0 - \epsilon, y_0 + \epsilon)$ such that every solution y with $y(0) \in I$ satisfies

$$\lim_{t\to\infty} y(t) = y_0.$$

Definition

The equilibrium solution y_0 is *asymptotically unstable* iff there exists $I = (y_0 - \epsilon, y_0 + \epsilon)$ such that for every solution y with $y(0) \in I$ holds

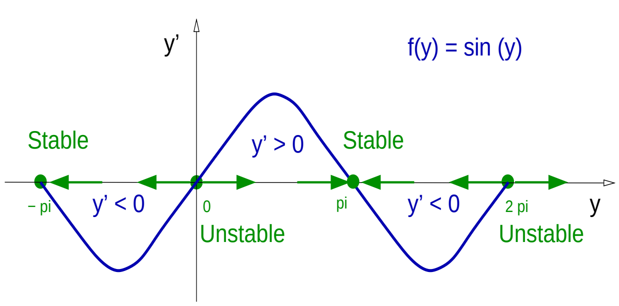
$$\lim_{t\to\infty} y(t) \neq y_0.$$

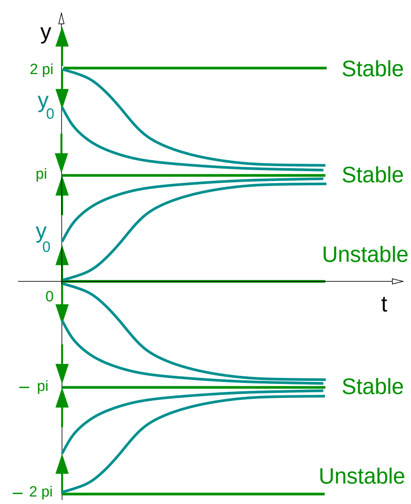
Equilibrium solutions and stability

Example

Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

Solution: Summary:





Autonomous systems (Sect. 2.5).

- Definition and examples.
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- **Population growth equation.**

Population growth equation (Logistic equation)

Example

Sketch a qualitative graph of solutions for different initial data conditions $y(0) = y_0$ to the population growth equation $y' = r\left(1 - \frac{y}{K}\right) y$, where r and K are given positive constants.

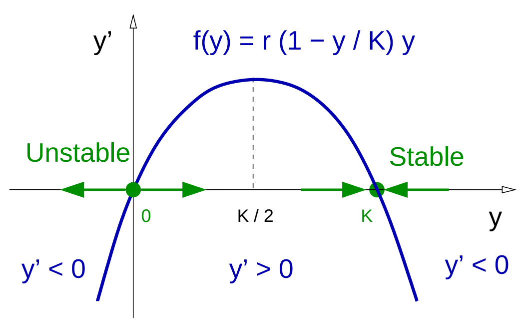
Solution:

(1) Plot the function

$$f(y) = r\left(1 - \frac{y}{K}\right) y.$$

(2) Find the zeros of f.

$$y_0 = 0, \quad y_0 = K.$$



The constants $y_0 = 0$ and $y_0 = K$ are the equilibrium solutions.

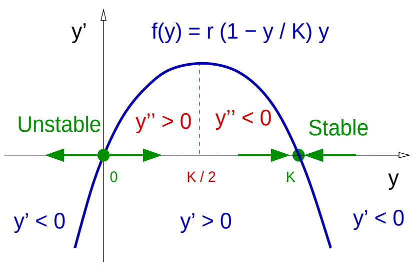
The solution y_0 is unstable, while $y_0 = K$ is stable.

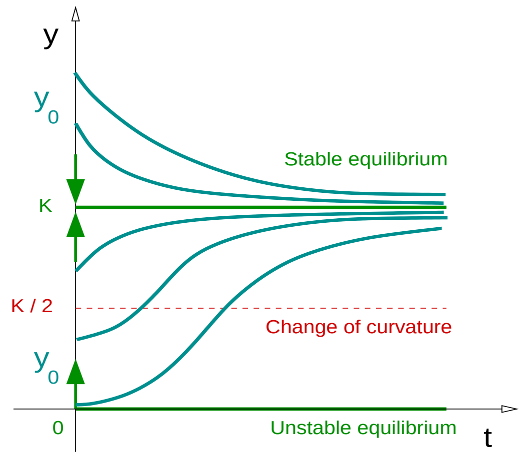
Population growth equation (Logistic equation)

Example

Sketch a qualitative graph of solutions for different initial data conditions $y(0) = y_0$ to the population growth equation $y' = r\left(1 - \frac{y}{K}\right) y$, where r and K are given positive constants.

Solution:





(3) For $y_0 \in (0, K)$ the solution is Increasing.

For $y_0 \in (K, \infty)$ the solution is Decreasing.

Population growth equation (Logistic equation)

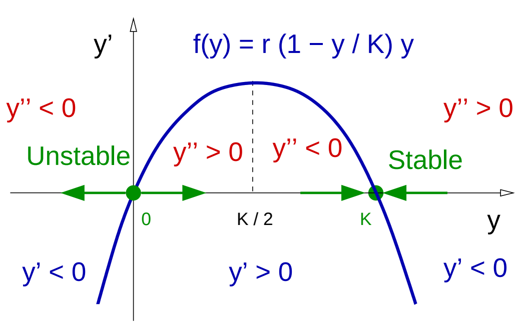
Remark: The curvature of the solution y depends on $f'(y)\, f(y)$.

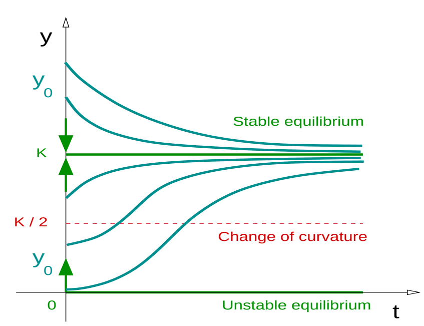
Theorem

If the function y is a solution of the autonomous system $y' = f(y)$, then the graph of y has positive curvature iff $f'(y)\, f(y) > 0$, and negative curvature iff $f'(y)\, f(y) < 0$.

Proof:

$$\frac{d^2y}{dt^2} = \frac{df}{dy}(y)\,\frac{dy}{dt}, \qquad \frac{dy}{dt} = f(y) \quad \Rightarrow \quad y'' = f'(y)\, f(y). \qquad \square.$$





Population growth equation (Logistic equation)

Example

Find the exact expression for the solutions to the population growth equation $y' = r\left(1 - \frac{y}{K}\right) y$, with $y(0) = y_0$.

Solution: This is a separable equation,

$$\frac{K}{r} \int \frac{y'\,dt}{(K - y)y} = t + c_0.$$

Substitution: $u = y(t)$, then $du = y'\,dt$,

$$\frac{K}{r} \int \frac{du}{(K - u)u} = t + c_0.$$

Partial fraction decomposition:

$$\frac{K}{r} \int \frac{1}{K} \Big[\frac{1}{(K - u)} + \frac{1}{u}\Big]\,du = t + c_0.$$

Population growth equation (Logistic equation)

Example

Find the exact expression for the solutions to the population growth equation $y' = r\left(1 - \frac{y}{K}\right) y$, with $y(0) = y_0$.

Solution: $\frac{K}{r} \int \frac{1}{K} \Big[\frac{1}{(K - u)} + \frac{1}{u}\Big]\,du = t + c_0.$

$$\big[-\ln(|K - y|) + \ln(|y|)\big] = rt + rc_0.$$

$$\ln\left(\frac{|y|}{|K - y|}\right) = rt + rc_0 \quad \Rightarrow \quad \frac{y}{K - y} = c\,e^{rt}, \quad c = e^{rc_0}.$$

$$y(t) = \frac{cK\,e^{rt}}{1 + c\,e^{rt}}, \quad c = \frac{y_0}{K - y_0}$$

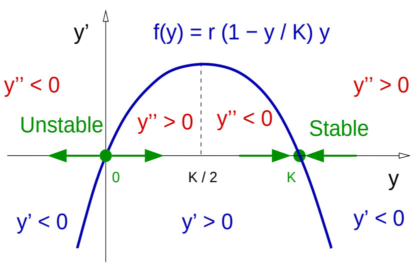
We conclude that $y(t) = \dfrac{Ky_0}{y_0 + (K - y_0)\,e^{-rt}}$. $\triangleleft$

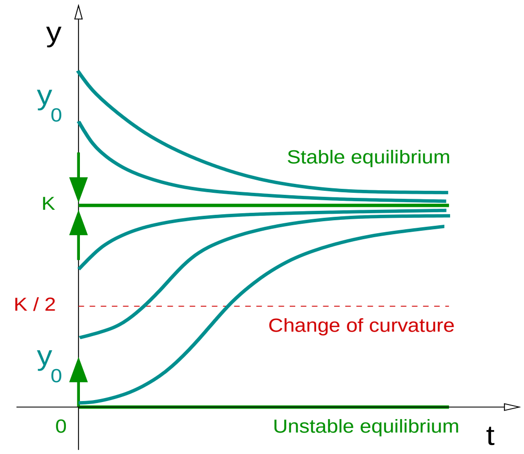
Population growth equation (Logistic equation)

Example

Sketch a qualitative graph of solutions for different initial data conditions $y(0) = y_0$ to the population growth equation $y' = r\left(1 - \frac{y}{K}\right) y$, where r and K are given positive constants.

Solution:





The solution is $y(t) = \dfrac{K y_0}{y_0 + (K - y_0)\, e^{-rt}}$.