

## Modeling with first order equations (Sect. 2.3).

- ▶ Main example: Salt in a water tank.
  - ▶ The experimental device.
  - ▶ The main equations.
  - ▶ Analysis of the mathematical model.
  - ▶ Predictions for particular situations.

### Salt in a water tank.

**Problem:** Describe the salt concentration in a tank with water if salty water comes in and goes out of the tank.

**Main ideas of the test:**

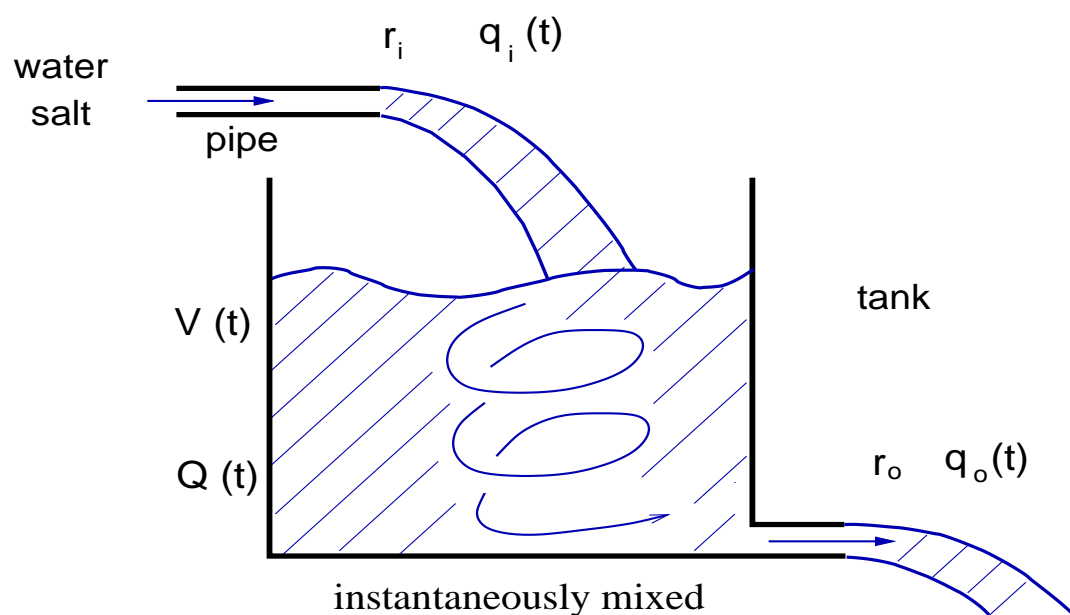
- ▶ Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- ▶ The amount of salt in the tank depends on the salt concentration coming in and going out of the tank.
- ▶ The salt in the tank also depends on the water rates coming in and going out of the tank.
- ▶ To construct a model means to find the differential equation that takes into account the above properties of the system.
- ▶ Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.

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### The experimental device.



## The experimental device.

### Definitions:

- ▶  $r_i(t)$ ,  $r_o(t)$ : Rates in and out of water entering and leaving the tank at the time  $t$ .
- ▶  $q_i(t)$ ,  $q_o(t)$ : Salt concentration of the water entering and leaving the tank at the time  $t$ .
- ▶  $V(t)$ : Water volume in the tank at the time  $t$ .
- ▶  $Q(t)$ : Salt mass in the tank at the time  $t$ .

### Units:

$$[r_i(t)] = [r_o(t)] = \frac{\text{Volume}}{\text{Time}}, \quad [q_i(t)] = [q_o(t)] = \frac{\text{Mass}}{\text{Volume}}.$$

$$[V(t)] = \text{Volume}, \quad [Q(t)] = \text{Mass}.$$

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## The main equations.

**Remark:** The mass conservation provides the main equations of the mathematical description for salt in water.

Main equations:

$$\frac{d}{dt} V(t) = r_i(t) - r_o(t), \quad \text{Volume conservation,} \quad (1)$$

$$\frac{d}{dt} Q(t) = r_i(t) q_i(t) - r_o(t) q_o(t), \quad \text{Mass conservation,} \quad (2)$$

$$q_o(t) = \frac{Q(t)}{V(t)}, \quad \text{Instantaneously mixed,} \quad (3)$$

$$r_i, r_o : \quad \text{Constants.} \quad (4)$$

## The main equations.

Remarks:

$$\left[ \frac{dV}{dt} \right] = \frac{\text{Volume}}{\text{Time}} = [r_i - r_o],$$

$$\left[ \frac{dQ}{dt} \right] = \frac{\text{Mass}}{\text{Time}} = [r_i q_i - r_o q_o],$$

$$[r_i q_i - r_o q_o] = \frac{\text{Volume}}{\text{Time}} \frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}.$$

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## Analysis of the mathematical model.

Eqs. (4) and (1) imply

$$V(t) = (r_i - r_o) t + V_0, \quad (5)$$

where  $V(0) = V_0$  is the initial volume of water in the tank.

Eqs. (3) and (2) imply

$$\frac{d}{dt} Q(t) = r_i q_i(t) - r_o \frac{Q(t)}{V(t)}. \quad (6)$$

Eqs. (5) and (6) imply

$$\frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \quad (7)$$

## Analysis of the mathematical model.

Recall:  $\frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t).$

Notation:  $a(t) = \frac{r_o}{(r_i - r_o) t + V_0},$  and  $b(t) = r_i q_i(t).$

The main equation of the description is given by

$$Q'(t) = -a(t) Q(t) + b(t).$$

Linear ODE for  $Q$ . Solution: Integrating factor method.

$$Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b(s) ds \right]$$

with  $Q(0) = Q_0$ , where  $\mu(t) = e^{A(t)}$  and  $A(t) = \int_0^t a(s) ds.$

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## Predictions for particular situations.

### Example

Assume that  $r_i = r_o = r$  and  $q_i$  are constants.

If  $r$ ,  $q_i$ ,  $Q_0$  and  $V_0$  are given, find  $Q(t)$ .

**Solution:** Always holds  $Q'(t) = -a(t) Q(t) + b(t)$ .

In this case:

$$a(t) = \frac{r_o}{(r_i - r_o)t + V_0} \Rightarrow a(t) = \frac{r}{V_0} = a_0,$$

$$b(t) = r_i q_i(t) \Rightarrow b(t) = r q_i = b_0.$$

We need to solve the IVP:

$$Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0.$$

## Predictions for particular situations.

### Example

Assume that  $r_i = r_o = r$  and  $q_i$  are constants.

If  $r$ ,  $q_i$ ,  $Q_0$  and  $V_0$  are given, find  $Q(t)$ .

**Solution:** Recall the IVP:  $Q'(t) = -a_0 Q(t) + b_0$ ,  $Q(0) = Q_0$ .

Integrating factor method:

$$A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad Q(t) = \frac{1}{\mu(t)} \left[ Q_0 + \int_0^t \mu(s) b_0 ds \right].$$

$$\int_0^t \mu(s) b_0 ds = \frac{b_0}{a_0} (e^{a_0 t} - 1) \Rightarrow Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right].$$

$$\text{So: } Q(t) = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}. \quad \text{But } \frac{b_0}{a_0} = r q_i \frac{V_0}{r} = q_i V_0.$$

We conclude:  $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$ .

## Predictions for particular situations.

### Example

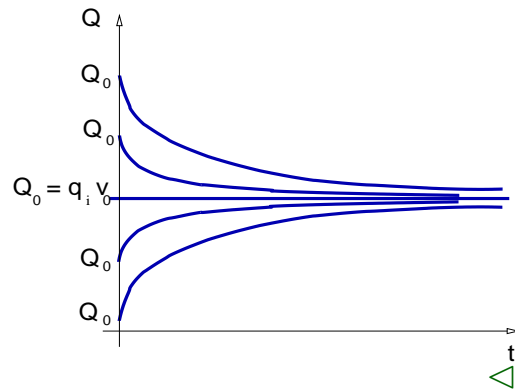
Assume that  $r_i = r_o = r$  and  $q_i$  are constants.

If  $r$ ,  $q_i$ ,  $Q_0$  and  $V_0$  are given, find  $Q(t)$ .

**Solution:** Recall:  $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$ .

Particular cases:

- ▶  $\frac{Q_0}{V_0} > q_i$ ;
- ▶  $\frac{Q_0}{V_0} = q_i$ , so  $Q(t) = Q_0$ ;
- ▶  $\frac{Q_0}{V_0} < q_i$ .



## Predictions for particular situations.

### Example

Assume that  $r_i = r_o = r$  and  $q_i$  are constants.

If  $r = 2$  liters/min,  $q_i = 0$ ,  $V_0 = 200$  liters,  $Q_0/V_0 = 1$  grams/liter, find  $t_1$  such that  $q(t_1) = Q(t_1)/V(t_1)$  is 1% the initial value.

**Solution:** This problem is a particular case  $q_i = 0$  of the previous Example. Since  $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$ , we get

$$Q(t) = Q_0 e^{-rt/V_0}.$$

Since  $V(t) = (r_i - r_o) t + V_0$  and  $r_i = r_o$ , we obtain  $V(t) = V_0$ .

So  $q(t) = Q(t)/V(t)$  is given by  $q(t) = \frac{Q_0}{V_0} e^{-rt/V_0}$ . Therefore,

$$\frac{1}{100} \frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0} e^{-rt_1/V_0} \Rightarrow e^{-rt_1/V_0} = \frac{1}{100}.$$



## Predictions for particular situations.

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**Solution:** Recall:  $e^{-rt_1/V_0} = \frac{1}{100}$ . Then,

$$-\frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right) = -\ln(100) \Rightarrow \frac{r}{V_0} t_1 = \ln(100).$$

We conclude that  $t_1 = \frac{V_0}{r} \ln(100)$ .

In this case:  $t_1 = 100 \ln(100)$ .

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## Predictions for particular situations.

### Example

Assume that  $r_i = r_o = r$  are constants. If  $r = 5 \times 10^6$  gal/year,  $q_i(t) = 2 + \sin(2t)$  grams/gal,  $V_0 = 10^6$  gal,  $Q_0 = 0$ , find  $Q(t)$ .

**Solution:** Recall:  $Q'(t) = -a(t)Q(t) + b(t)$ . In this case:

$$a(t) = \frac{r_o}{(r_i - r_o)t + V_0} \Rightarrow a(t) = \frac{r}{V_0} = a_0,$$

$$b(t) = r_i q_i(t) \Rightarrow b(t) = r[2 + \sin(2t)].$$

We need to solve the IVP:  $Q'(t) = -a_0 Q(t) + b(t)$ ,  $Q(0) = 0$ .

$$Q(t) = \frac{1}{\mu(t)} \int_0^t \mu(s) b(s) ds, \quad \mu(t) = e^{a_0 t},$$

We conclude:  $Q(t) = re^{-rt/V_0} \int_0^t e^{rs/V_0} [2 + \sin(2s)] ds$ .