Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- Homogeneous equations.

Separable ODE.

Definition

Given functions $h, g : \mathbb{R} \to \mathbb{R}$, a first order ODE on the unknown function $y : \mathbb{R} \to \mathbb{R}$ is called *separable* iff the ODE has the form

$$h(y)\, y'(t) = g(t).$$

Remark:

A differential equation $y'(t) = f(t, y(t))$ is separable iff

$$y' = \frac{g(t)}{h(y)} \quad \Leftrightarrow \quad f(t, y) = \frac{g(t)}{h(y)}.$$

Notation:

In lecture: t, $y(t)$ and $h(y)\, y'(t) = g(t)$.

In textbook: x, $y(x)$ and $M(x) + N(y)\, y'(x) = 0$.

Therefore: $h(y) = N(y)$ and $g(t) = -M(t)$.

Separable ODE.

Example

Determine whether the differential equation below is separable,

$$y'(t) = \frac{t^2}{1 - y^2(t)}.$$

Solution: The differential equation is separable, since it is equivalent to

$$\left(1 - y^2\right) y'(t) = t^2 \quad \Rightarrow \quad \begin{cases} g(t) = t^2, \\ h(y) = 1 - y^2. \end{cases}$$

$\triangleleft$

Remark: The functions g and h are not uniquely defined. Another choice here is:

$$g(t) = c\, t^2, \quad h(y) = c\,(1 - y^2), \quad c \in \mathbb{R}.$$

Separable ODE.

Example

Determine whether The differential equation below is separable,

$$y'(t) + y^2(t)\, \cos(2t) = 0$$

Solution: The differential equation is separable, since it is equivalent to

$$\frac{1}{y^2}\, y'(t) = -\cos(2t) \quad \Rightarrow \quad \begin{cases} g(t) = -\cos(2t), \\ h(y) = \dfrac{1}{y^2}. \end{cases}$$

$\triangleleft$

Remark: The functions g and h are not uniquely defined. Another choice here is:

$$g(t) = \cos(2t), \quad h(y) = -\frac{1}{y^2}.$$

Separable ODE.

Remark: Not every first order ODE is separable.

Example

- The differential equation $y'(t) = e^{y(t)} + \cos(t)$ is not separable.
- The linear differential equation $y'(t) = -\frac{2}{t}\,y(t) + 4t$ is not separable.
- The linear differential equation $y'(t) = -a(t)\,y(t) + b(t)$, with $b(t)$ non-constant, is not separable.

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Solutions to separable ODE.

Theorem (Separable equations)

If the functions $g, h : \mathbb{R} \to \mathbb{R}$ are continuous, with $h \neq 0$ and with primitives G and H, respectively; that is,

$$G'(t) = g(t), \qquad H'(u) = h(u),$$

then, the separable ODE

$$h(y)\, y' = g(t)$$

has infinitely many solutions $y : \mathbb{R} \to \mathbb{R}$ satisfying the algebraic equation

$$H(y(t)) = G(t) + c,$$

where $c \in \mathbb{R}$ is arbitrary.

Remark: Given functions g, h, find their primitives G, H.

Solutions to separable ODE.

Example

Find all solutions $y : \mathbb{R} \to \mathbb{R}$ to the ODE $y'(t) = \dfrac{t^2}{1 - y^2(t)}$.

Solution: The equation is equivalent to $\left(1 - y^2\right) y'(t) = t^2$.
Therefore, the functions g, h are given by

$$g(t) = t^2, \qquad h(u) = 1 - u^2.$$

Their primitive functions, G and H, respectively, are given by

$$g(t) = t^2 \quad \Rightarrow \quad G(t) = \frac{t^3}{3},$$

$$h(u) = 1 - u^2 \quad \Rightarrow \quad H(u) = u - \frac{u^3}{3}.$$

Then, the Theorem above implies that the solution y satisfies the algebraic equation

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c, \quad c \in \mathbb{R}. \qquad \triangleleft$$

Solutions to separable ODE.

Remarks:

- The equation $y(t) - \dfrac{y^3(t)}{3} = \dfrac{t^3}{3} + c$ is algebraic in y, since there is no y' in the equation.
- Every function y satisfying the algebraic equation

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c,$$

  is a solution of the differential equation above.
- We now verify the previous statement: Differentiate on both sides with respect to t, that is,

$$y'(t) - 3\left(\frac{y^2(t)}{3}\right) y'(t) = 3\,\frac{t^2}{3} \quad \Rightarrow \quad (1 - y^2)\, y' = t^2.$$

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Explicit and implicit solutions.

Remark:

The solution $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$ is given in implicit form.

Definition

Assume the notation in the Theorem above. The solution y of a separable ODE is given in *implicit form* iff function y is specified by

$$H\big(y(t)\big) = G(t) + c,$$

The solution y of a separable ODE is given in *explicit form* iff function H is invertible and y is specified by

$$y(t) = H^{-1}\big(G(t) + c\big).$$

Explicit and implicit solutions.

Example

Use the main idea in the proof of the Theorem above to find the solution of the IVP

$$y'(t) + y^2(t)\cos(2t) = 0, \qquad y(0) = 1.$$

Solution: The differential equation is separable, with

$$g(t) = -\cos(2t), \qquad h(y) = \frac{1}{y^2}.$$

The main idea in the proof of the Theorem above is this: integrate on both sides of the equation,

$$\frac{y'(t)}{y^2(t)} = -\cos(2t) \quad \Leftrightarrow \quad \int \frac{y'(t)}{y^2(t)}\,dt = -\int \cos(2t)\,dt + c.$$

The substitution $u = y(t)$, $du = y'(t)\,dt$, implies that

$$\int \frac{du}{u^2} = -\int \cos(2t)\,dt + c \quad \Leftrightarrow \quad -\frac{1}{u} = -\frac{1}{2}\sin(2t) + c.$$

Explicit and implicit solutions.

Example

Use the main idea in the proof of the Theorem above to find the solution of the IVP

$$y'(t) + y^2(t)\cos(2t) = 0, \qquad y(0) = 1.$$

Solution: Recall: $-\frac{1}{u} = -\frac{1}{2}\sin(2t) + c.$

Substitute the unknown function y back in the equation above,

$$-\frac{1}{y(t)} = -\frac{1}{2}\sin(2t) + c. \qquad \text{(Implicit form.)}$$

$$y(t) = \frac{2}{\sin(2t) - 2c}. \qquad \text{(Explicit form.)}$$

The initial condition implies that $1 = y(0) = \frac{2}{0 - 2c}$, so $c = -1$.

We conclude that $y(t) = \frac{2}{\sin(2t) + 2}$. ◁

Separable differential equations (Sect. 2.2).

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- **Homogeneous equations.**

Homogeneous equations.

Definition

The first order ODE $y'(t) = f\big(t, y(t)\big)$ is called *homogeneous* iff for every numbers $c, t, u \in \mathbb{R}$ the function f satisfies

$$f(ct, cu) = f(t, u).$$

Remark:

- The function f is invariant under the change of scale of its arguments.
- If $f(t, u)$ has the property above, it must depend only on u/t.
- So, there exists $F : \mathbb{R} \to \mathbb{R}$ such that $f(t, u) = F\Big(\dfrac{u}{t}\Big)$.
- Therefore, a first order ODE is homogeneous iff it has the form

$$y'(t) = F\Big(\frac{y(t)}{t}\Big).$$

Homogeneous equations.

Example

Show that the equation below is homogeneous,

$$(t - y)\, y' - 2y + 3t + \frac{y^2}{t} = 0.$$

Solution: Rewrite the equation in the standard form

$$(t - y)\, y' = 2y - 3t - \frac{y^2}{t} \quad \Rightarrow \quad y' = \frac{\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{(t - y)}.$$

Divide numerator and denominator by t. We get,

$$y' = \frac{\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{(t - y)} \, \frac{\Big(\dfrac{1}{t}\Big)}{\Big(\dfrac{1}{t}\Big)} \quad \Rightarrow \quad y' = \frac{2\Big(\dfrac{y}{t}\Big) - 3 - \Big(\dfrac{y}{t}\Big)^2}{\Big[1 - \Big(\dfrac{y}{t}\Big)\Big]}.$$

Homogeneous equations.

Example

Show that the equation below is homogeneous,

$$(t - y)\, y' - 2y + 3t + \frac{y^2}{t} = 0.$$

Solution: Recall: $y' = \dfrac{2\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right)^2}{\left[1 - \left(\frac{y}{t}\right)\right]}$.

We conclude that the ODE is homogeneous, because the right-hand side of the equation above depends only on y/t.

Indeed, in our case:

$$f(t, y) = \frac{2y - 3t - (y^2/t)}{t - y}, \qquad F(x) = \frac{2x - 3 - x^2}{1 - x},$$

and $f(t, y) = F(y/t)$. ◁

Homogeneous equations.

Example

Determine whether the equation below is homogeneous,

$$y' = \frac{t^2}{1 - y^3}.$$

Solution:

Divide numerator and denominator by t^3, we obtain

$$y' = \frac{t^2}{(1 - y^3)} \frac{\left(\frac{1}{t^3}\right)}{\left(\frac{1}{t^3}\right)} \quad \Rightarrow \quad y' = \frac{\left(\frac{1}{t}\right)}{\left(\frac{1}{t^3}\right) - \left(\frac{y}{t}\right)^3}.$$

We conclude that the differential equation is not homogeneous. ◁

Homogeneous equations.

Theorem

If the differential equation $y'(t) = f\big(t, y(t)\big)$ *is homogeneous, then the differential equation for the unknown* $v(t) = \dfrac{y(t)}{t}$ *is separable.*

Remark: Homogeneous equations can be transformed into separable equations.

Proof: If $y' = f(t, y)$ is homogeneous, then it can be written as $y' = F(y/t)$ for some function F. Introduce $v = y/t$. This means,

$$y(t) = t\,v(t) \quad \Rightarrow \quad y'(t) = v(t) + t\,v'(t).$$

Introducing all this into the ODE we get

$$v + t\,v' = F(v) \quad \Rightarrow \quad v' = \frac{\big(F(v) - v\big)}{t}.$$

This last equation is separable. □

Homogeneous equations.

Example

Find all solutions y of the ODE $y' = \dfrac{t^2 + 3y^2}{2ty}$.

Solution: The equation is homogeneous, since

$$y' = \frac{t^2 + 3y^2}{2ty} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad \Rightarrow \quad y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$

Therefore, we introduce the change of unknown $v = y/t$, so $y = t\,v$ and $y' = v + t\,v'$. Hence

$$v + t\,v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t\,v' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v}$$

We obtain the separable equation $v' = \dfrac{1}{t}\left(\dfrac{1 + v^2}{2v}\right)$.

Homogeneous equations.

Example

Find all solutions y of the ODE $y' = \dfrac{t^2 + 3y^2}{2ty}$.

Solution: Recall: $v' = \dfrac{1}{t}\Big(\dfrac{1+v^2}{2v}\Big)$. We rewrite and integrate it,

$$\frac{2v}{1+v^2}\, v' = \frac{1}{t} \quad \Rightarrow \quad \int \frac{2v}{1+v^2}\, v'\, dt = \int \frac{1}{t}\, dt + c_0.$$

The substitution $u = 1 + v^2(t)$ implies $du = 2v(t)\, v'(t)\, dt$, so

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0 \quad \Rightarrow \quad \ln(u) = \ln(t) + c_0 \quad \Rightarrow \quad u = e^{\ln(t)+c_0}.$$

But $u = e^{\ln(t)} e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. Hence

$$1 + v^2 = c_1 t \quad \Rightarrow \quad 1 + \Big(\frac{y}{t}\Big)^2 = c_1 t \quad \Rightarrow \quad y(t) = \pm t\sqrt{c_1 t - 1}.$$