

Name: Key ID Number: _____

TA: _____ Section Time: _____

MTH 235
Exam 3 Makeup
April 15, 2010
50 minutes
Sects: 6.1-6.6,
7.1-7.6, 7.8.

No notes. No books. No Calculators.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem,
the worst answer will be graded.
Show all your work. Box your answers.

1. (20 points) Use the Laplace transform to find the solution y to the initial value problem

$$y'' + 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 3$$

$$\begin{aligned} \mathcal{L}[y''] + 2\mathcal{L}[y'] - 3\mathcal{L}[y] &= 0 & \mathcal{L}[y''] &= s^2 \mathcal{L}[y] - s - 3 \\ \mathcal{L}[y'] &= s \mathcal{L}[y] - 1 & & \end{aligned} \Rightarrow$$

$$(s^2 + 2s - 3) \mathcal{L}[y] - s - 3 + 2(-1) = 0$$

$$(s^2 + 2s - 3) \mathcal{L}[y] = s + 5 \Rightarrow \mathcal{L}[y] = \frac{s+5}{(s^2+2s-3)}$$

$$s^2 + 2s - 3 = 0 \Rightarrow s = \frac{-2 \pm \sqrt{4+12}}{2} = \frac{-2 \pm 4}{2} \Rightarrow s_+ = 1, \quad s_- = -3$$

$$s^2 + 2s - 3 = (s+3)(s-1) \Rightarrow \mathcal{L}[y] = \frac{(s+5)}{(s+3)(s-1)}$$

$$\frac{(s+5)}{(s+3)(s-1)} = \frac{a}{(s+3)} + \frac{b}{(s-1)} = \frac{a(s-1) + b(s+3)}{(s+3)(s-1)} = \frac{(a+b)s + (-a+3b)}{(s+3)(s-1)}$$

$$\begin{aligned} a+b &= 1 \Rightarrow b = 1-a \\ -a+3b &= 5 \end{aligned} \Rightarrow -a + 3(1-a) = 5 \Rightarrow -a + 3 - 3a = 5 \Rightarrow -4a = 2$$

$$\left[a = -\frac{1}{2} \right] \left[b = \frac{3}{2} \right] \Rightarrow \mathcal{L}[y] = -\frac{1}{2} \left(\frac{1}{s+3} \right) + \frac{3}{2} \left(\frac{1}{s-1} \right) \Rightarrow$$

$$\Rightarrow \mathcal{L}[y] = -\frac{1}{2} \mathcal{L}[e^{-3t}] + \frac{3}{2} \mathcal{L}[e^t] \Rightarrow \boxed{y = \frac{1}{2} (3e^t - e^{-3t})}$$

2. (20 points) Use the Laplace transform to find the solution y to the initial value problem

$$y'' + 2y = -3\delta(t-5), \quad y(0) = 0 \quad y'(0) = 2$$

$$\mathcal{L}[y''] + 2\mathcal{L}[y] = -3\mathcal{L}[\delta(t-5)] \quad , \quad \mathcal{L}[y''] = s^2 \mathcal{L}[y] - 2$$

$$(s^2 + 2) \mathcal{L}[y] - 2 = -3 e^{-5s} \Rightarrow \mathcal{L}[y] = \frac{2}{(s^2+2)} - \frac{3 e^{-5s}}{(s^2+2)}$$

$$\mathcal{L}[y] = \sqrt{2} \mathcal{L}[\sin(\sqrt{2}t)] - \frac{3}{\sqrt{2}} e^{-5s} \mathcal{L}[\sin(\sqrt{2}t)]$$

$$\mathcal{L}[y] = \sqrt{2} \mathcal{L}[\sin(\sqrt{2}t)] - \frac{3}{\sqrt{2}} \mathcal{L}[u(t-5) \sin(\sqrt{2}(t-5))]$$

$$y = \sqrt{2} \sin(\sqrt{2}t) - \frac{3}{\sqrt{2}} u(t-5) \sin(\sqrt{2}(t-5))$$

3. (15 points) Use convolutions to express the function f having the Laplace transform

$$\mathcal{L}[f(t)] = \frac{1}{s^5 [(s-3)^2 - 4]}.$$

$$\mathcal{L}[f] = \left(\frac{1}{s^5} \right) \left(\frac{1}{(s-3)^2 - 4} \right)$$

$$= \frac{1}{4!} \left(\frac{4!}{s^5} \right) \quad \frac{1}{2} \left(\frac{2}{(s-3)^2 - 4} \right)$$

$$\mathcal{L}[f] = \frac{1}{4!} \mathcal{L}[t^4] \quad \frac{1}{2} \mathcal{L}[e^{3t} \sinh(2t)]$$

$$f(t) = \frac{1}{2(4!)} \int_0^t (t-\tau)^4 e^{3\tau} \sin(2\tau) d\tau$$

4. (a) (20 points) Find the general solution \mathbf{x} to the 2×2 linear system

$$\mathbf{x}'(t) = A \mathbf{x}(t), \quad A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}.$$

(b) (5 points) Sketch a qualitative phase portrait of the solution trajectories.

$$(a) P(\lambda) = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-2) + 5 = \lambda^2 - 4 + 5 = \lambda^2 + 1 \quad (=)$$

$$\lambda^2 + 1 = 0 \Rightarrow \boxed{\lambda_{\pm} = \pm i}$$

$$\lambda = i \Rightarrow A - \lambda I = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \xrightarrow{+2+i} \begin{bmatrix} 5 & -5(2+i) \\ 1 & -(2+i) \end{bmatrix} \rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & -(2+i) \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = (2+i)v_2$$

$$\underline{v}^{(+)} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \quad \lambda_+ = i$$

$$\underline{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

$\underline{a} \quad \underline{b}$

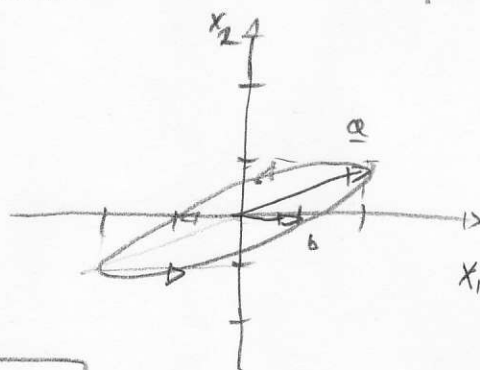
$$\underline{v}^{(-)} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix} \quad \lambda_- = -i$$

$$\underline{x}^{(1)} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(t)$$

$$\underline{x}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sin(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t)$$

$$\underline{x}(t) = c_1 \begin{bmatrix} 2 \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} 2 \sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$$

(b)



5. (20 points) Find the solution \mathbf{x} to the initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}.$$

$$P(\lambda) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (\lambda+9)(\lambda-3) + 36 = \lambda^2 + 4\lambda - 3\lambda - 27 + 36$$

$$\lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda_{\pm} = \frac{-6 \pm \sqrt{36 - 36}}{2} = -3.$$

$$\lambda_{\pm} = -3$$

$$A + 3I = \begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 3v_2$$

$$v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \lambda = -3$$

$$(A + 3I)w = v$$

$$\left[\begin{array}{cc|c} 6 & -18 & 3 \\ 2 & -6 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 2 & -6 & 1 \\ 2 & -6 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 2 & -6 & 1 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow 2w_1 = 6w_2 + 1$$

$$w = \begin{bmatrix} 3 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \quad w_2 = 0 \Rightarrow w = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$x(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right) e^{-3t}$$

#	Pts	Score
1	20	
2	20	
3	15	
4	25	
5	20	
Σ	100	

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -2 \begin{bmatrix} 0 & -1/2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (-2) \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$x(t) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} - 2 \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right) e^{-3t}$$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > a $
8. $\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > a $
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
11. $t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau) d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$