101'

Name: key ID Number: 34' ✓

TA: Please verify! Section Time: ______

MTH 235
Exam 3
April 13, 2010
50 minutes
Sects: 6.1-6.6, 7.1-7.6, 7.8.

No notes. No books. No Calculators.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. $\boxed{\textit{Box your answers.}}$

1. (20 points) Use the Laplace transform to find the solution y to the initial value problem

$$y'' + 3y' + 2y = 0, \qquad y(0) = \cancel{4}\ 0, \quad y'(0) = 3.$$

$$\mathcal{L}[y''] + 3\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = 0$$

+3 pts

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) = s^2\,\mathcal{L}[y] - 3$$

$$\mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0) = s\,\mathcal{L}[y]$$

+3 pts

$$(s^2 + 3s + 2)\,\mathcal{L}[y] - 3 = 0$$

$$(s^2 + 3s + 2)\,\mathcal{L}[y] = 3 \Rightarrow \boxed{\mathcal{L}[y] = \frac{3}{(s^2+3s+2)}}$$

+3 pts

$$s^2 + 3s - 2 = 0 \Rightarrow s_{\pm} = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2} = \begin{cases} -1 \\ -2. \end{cases}$$

$$\boxed{s^2 + 3s + 2 = (s+1)(s+2)} \Rightarrow \boxed{\mathcal{L}[y] = \frac{3}{(s+1)(s+3)}}$$

a

$$\frac{3}{(s+1)(s+2)} = \frac{a}{(s+1)} + \frac{b}{(s+2)} = \frac{a(s+2)+b(s+1)}{(s+1)(s+2)}$$

$$3 = (a+b)s + (2a+b) \Rightarrow \left.\begin{array}{l} a+b=0 \Rightarrow b=-a \\ 2a+b=3 \end{array}\right] \Rightarrow 2a-a=3$$

$$\boxed{a=3}$$
$$\boxed{b=-3}$$

+6 pts

$$\mathcal{L}[y] = \frac{3}{(s+1)} - \frac{3}{(s+2)} = 3\,\mathcal{L}[e^{-t}] - 3\,\mathcal{L}[e^{-2t}]$$

$$\boxed{y(t) = 3\left(e^{-t} - e^{-2t}\right)}$$

+5 pts

5'

2. (20 points) Use the Laplace transform to find the solution y to the initial value problem

$$y'' + 2y = -2\delta(t-3), \qquad y(0) = 0, \qquad y'(0) = 0.$$

$$\mathcal{L}[y''] + 2\mathcal{L}[y] = -2\,\mathcal{L}[\delta(t-3)]$$

+ 3pts

$$(s^2 + 2)\,\mathcal{L}[y] = -2\,e^{-3s} \Rightarrow \boxed{\mathcal{L}[y] = -2\,\frac{e^{-3s}}{s^2+2}}$$

+6 pts

$$\mathcal{L}[y] = -\sqrt{2}\,e^{-3s}\,\frac{\sqrt{2}}{s^2+2} = -\sqrt{2}\,e^{-3s}\,\mathcal{L}[\sin(\sqrt{2}\,t)]$$

+5 pts

$$= -\sqrt{2}\,\mathcal{L}\left[u(t-3)\,\sin\left[\sqrt{2}\,(t-3)\right]\right]$$

$$\boxed{y(t) = -\sqrt{2}\,u(t-3)\,\sin\left[\sqrt{2}\,(t-3)\right]}$$

+ 6 pts

15

3. (~~20~~ points) Use convolutions to express the function f that has the Laplace Transform

$$\mathcal{L}[f(t)] = \frac{1}{(s^2+3)(s^2-4)}.$$

$$\mathcal{L}[f] = \frac{1}{(s^2+3)} \quad \frac{1}{(s^2-4)} = \left(\frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(s^2+3)}\right)\left(\frac{1}{2} \frac{2}{(s^2-4)}\right)$$

+5 pts

$$\mathcal{L}[f] = \mathcal{L}\left[\frac{1}{\sqrt{3}} \sin(\sqrt{3}\, t)\right] \mathcal{L}\left[\frac{1}{2} \sinh(2t)\right]$$

+5 pts

$$\boxed{f(t) = \int_0^t \frac{1}{2\sqrt{3}} \sin(\sqrt{3}\,(t-\tau)) \sinh(2\tau)\, d\tau}$$

+5pts

or

$$\boxed{f(t) = \int_0^t \frac{1}{2\sqrt{3}} \sinh(2(t-\tau)) \sin(\sqrt{3}\,\tau)\, d\tau}$$

4. (20 points) Find the general solution $\boldsymbol{x}$ to the 2×2 linear system
(a)

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t), \qquad A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}.$$

(b) (5 points) Sketch a qualitative phase portrait of the solution trajectories.

(a) $p(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-1) - 4 = \lambda^2 + 2\lambda - \lambda - 2 - 4$

$p(\lambda) = \lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda_\pm = \dfrac{-1 \pm \sqrt{1+24}}{2} = \dfrac{-1 \pm 5}{2}$

$\boxed{\lambda_+ = 2}$ $\boxed{\lambda_- = -3}$ $\boxed{\lambda_- < 0 < \lambda_+}$ +5 pts

$\lambda_+ = 2$

$A - 2I = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = v_2 \Rightarrow \boxed{\underline{v}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \lambda_+ = 2}$

+5pts

$\lambda_- = -3$

$A + 3I = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 4v_1 = v_2 \Rightarrow \boxed{\underline{v}^{(-)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \ \lambda_- = -3}$

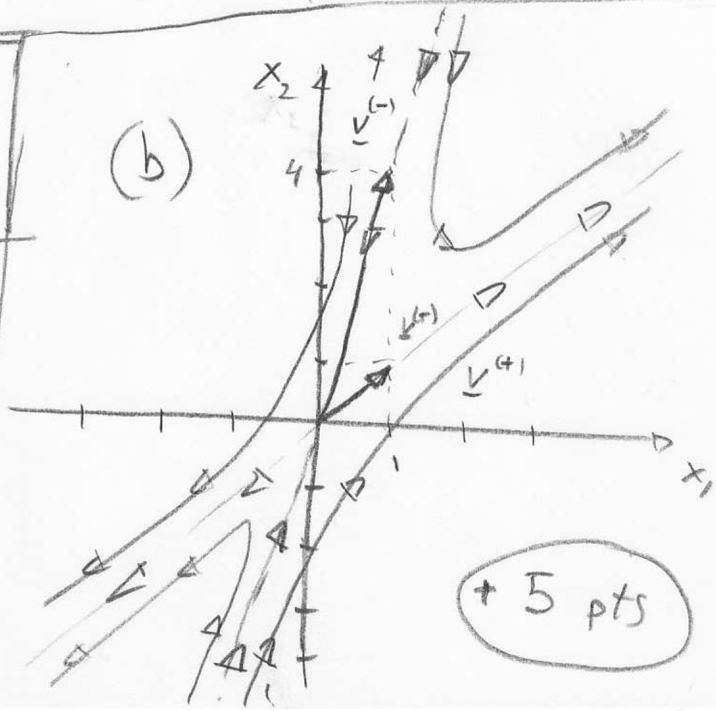
$v_2 = 4, \ v_1 = 1$

+5 pts

$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-3t}}$

+5 pts

(b)



+5 pts

8'

5. (20 points) Find the solution x to the initial value problem

$$x'(t) = A\,x(t), \qquad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

$$P(\lambda) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda-1) + 1 = \lambda^2 - \lambda - 3\lambda + 3 + 1$$

$$P(\lambda) = \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_\pm = \frac{+4 \pm \sqrt{16-16}}{2} = 2.$$

+4pts

$\boxed{\lambda = 2}$ repeated.

$$A - 2I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \boxed{\underline{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \lambda = 2}$$

+4 pts

$\underline{w}$ sol. of $(A - 2I)\underline{w} = \underline{v}$

$$\left[\begin{array}{cc|c} -1 & -1 & -1 \\ 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow w_1 = -w_2 + 1 \Rightarrow \underline{w} = \begin{bmatrix} -w_2 + 1 \\ w_2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \underline{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{choose } w_2 = 0 \Rightarrow \boxed{\underline{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

+4pts

$$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{2t}}$$

+3pts

#	Pts	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Σ	100	

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} c_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} c_2 \Rightarrow$$

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{0-1} \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{matrix} 2 \\ 1 \end{matrix}$$

$$\boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}} \Rightarrow \boxed{\underline{x}(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + 3 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{2t}}$$

+5pts

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$