Name: Key ID Number: extra paper needed

TA: Please verify Section Time: ______

MTH 235
Exam 2
March 2, 2010
50 minutes
Sects: 3.1-3.6,
5.2, 5.4, 5.5.

No notes. No books. No Calculators.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. *Box your answers.*

1. (15 points) Find the solution to the initial value problem

$$y'' + y' - 6y = 0, \qquad y(0) = 3, \qquad y'(0) = -4.$$

$r^2 + r - 6 = 0$ $\qquad r = \dfrac{-1 \pm \sqrt{1+24}}{2} = \dfrac{-1 \pm 5}{2}$

$\boxed{r_1 = 2}$ $\quad \boxed{r_2 = -3}$

5 pts

$y(t) = c_1 e^{2t} + c_2 e^{-3t}$

$y'(t) = 2c_1 e^{2t} - 3c_2 e^{-3t}$

5 pts

$3 = y(0) = c_1 + c_2 \quad \Rightarrow \quad c_2 = 3 - c_1$

$-4 = y'(0) = 2c_1 - 3c_2$

$2c_1 - 3(3 - c_1) = -4$

$2c_1 - 9 + 3c_1 = -4$

$5c_1 = 5 \Rightarrow \boxed{c_1 = 1}$ $\quad \boxed{c_2 = 2}$

$\boxed{y(t) = e^{2t} + 2e^{-3t}}$

5pts

2. (~~18~~ 21 points) Find the indicial equation and the recurrence relation for the coefficients of a power series solution near the regular singular point $x_0 = 0$ to the equation

$$2x^2\, y'' + 3x\, y' + \left(2x^2 - 1\right) y = 0.$$

Also find the first three terms of a power series solution y.

$x_0 = 0$: regular singular point

$$y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r)}, \quad y'(x) = \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{(n+r-1)}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{(n+r-2)}$$

2pts

$$(2x^2-1)\, y = \sum_{n=0}^{\infty} 2\, a_n\, x^{(n+r+2)} + \sum_{n=0}^{\infty} (-a_n)\, x^{n+r}$$

$m = n+2$, $m \to n$

$$(2x^2-1)\, y = \sum_{n=2}^{\infty} 2\, a_{n-2}\, x^{(n+r)} + \sum_{n=0}^{\infty} (-a_n)\, x^{(n+r)}$$

4pts

$$3x\, y' = \sum_{n=0}^{\infty} 3(n+r)\, a_n\, x^{(n+r)}$$

$$2x^2\, y'' = \sum_{n=0}^{\infty} 2(n+r)(n+r-1)\, a_n\, x^{n+r}$$

2 pts

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)\, a_n x^{(n+r)} + \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{(n+r)} + \sum_{n=2}^{\infty} 2\, a_{n-2}\, x^{(n+r)}$$

$$+ \sum_{n=0}^{\infty} (-a_n)\, x^{(n+r)} = 0$$

$$2r(r-1)\, a_0 x^{r} + 2(r+1)r\, a_1 x^{r+1} + 3r\, a_0 x^{r} + 3(r+1)\, a_1 x^{(r+1)}$$

$$- a_0 x^{r} - a_1 x^{(r+1)}$$

$$+ \sum_{n=0}^{\infty} \left[2(n+r)(n+r-1)\, a_n + 3(n+r)\, a_n - a_n + 2\, a_{n-2} \right] x^{(n+r)} = 0$$

2pts

$$\boxed{2r(r-1) + 3r - 1 = 0} \Rightarrow \boxed{2r^2 + r - 1 = 0} \quad r = \frac{-1 \pm \sqrt{1+8}}{4}$$

$$\boxed{2(r+1)r + 3(r+1) - 1 = 0}$$

$$r = \frac{-1 \pm 3}{4} \nearrow \tfrac{1}{2},\ \searrow -1$$

$$\boxed{r_1 = \frac{1}{2}}$$

$$\boxed{\left[2(n+r)(n+r-1) + 3(n+r) - 1 \right] a_n + 2\, a_{n-2} = 0}$$

4pts

$$\left[2\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right) + 3\left(n+\tfrac{1}{2}\right) - 1 \right] a_n + 2\, a_{n-2} = 0$$

Choose $\boxed{r_1 = \frac{1}{2}}$

$$\left(2\left(n^2 - \tfrac{1}{4}\right) + 3n + \tfrac{1}{2} \right) a_n + 2\, a_{n-2} = 0$$

$$\boxed{\left(2n^2 + 3n\right) a_n + 2\, a_{n-2} = 0}$$

2pts

$$a_n = -\frac{2\, a_{n-2}}{2n^2+3n} \iff \boxed{a_n = \frac{-2\, a_{n-2}}{n(2n+3)}}$$

Since $2(r_1+1)r_1 + 3(r_1+1) - 1 \neq 0 \Rightarrow a_1 = 0 \Rightarrow \boxed{a_{2k+1} = 0}$

$n=2 \Rightarrow a_2 = -\frac{2\, a_0}{2(7)} \Rightarrow \boxed{a_2 = -\frac{a_0}{7}}$

$n=4 \Rightarrow a_4 = -\frac{2\, a_2}{4(11)} = -\frac{2}{4}\frac{1}{11}\left(-\frac{1}{7}\right)a_0 \Rightarrow \boxed{a_4 = \frac{a_0}{(11)(7)2}}$

$Y(x) = x^{r_1}\left(a_0 + a_2 x^2 + a_4 x^4 + \dots\right)$

3pts

$$\boxed{Y(x) = a_0\, x^{1/2}\left[1 - \frac{1}{7}x^2 + \frac{1}{(2)(7)(11)}x^4 + \dots\right]}$$

2pts

3. (17 points) Find a particular solution to the non-homogeneous equation

$$t\,y'' - (1+t)\,y' + y = t^2 e^{2t},$$

knowing that $y_1(t) = (1+t)$ and $y_2(t) = e^t$ are solutions to the homogeneous equation.

$$y'' - \frac{(1+t)}{t} y' + \frac{y}{t} = t e^{2t} \Rightarrow \boxed{g(t) = t e^{2t}} \quad \text{2pts}$$

$$W = \begin{vmatrix} 1+t & 1 \\ e^t & e^t \end{vmatrix} = (1+t)e^t - e^t \Rightarrow \boxed{W = t e^t} \quad \text{4pts}$$

$$u_1' = -\frac{Y_2\, g}{W} = -\frac{e^t \, t e^{2t}}{t e^t} = -e^{2t} \Rightarrow \boxed{u_1 = -\frac{e^{2t}}{2}} \quad \text{4pts}$$

$$u_2' = \frac{Y_1\, g}{W} = \frac{(1+t)\, t e^{2t}}{t e^t} = (1+t)\, e^t$$

$$u_2 = e^t + \int t e^t dt = e^t + t e^t - \int e^t dt = t e^t \Rightarrow \boxed{u_2 = t e^t} \quad \text{4pts}$$

$$Y_p = u_1 Y_1 + u_2 Y_2 = -\frac{e^{2t}}{2}(1+t) + t e^t e^t$$

$$Y_p = e^{2t}\left(-\frac{1}{2} - \frac{t}{2} + t\right) \Rightarrow \boxed{Y_p = \frac{1}{2}(t-1)\, e^{2t}}$$

3pts

4. (17 points) Find the general solution to the differential equation

$$y'' - 6y' + 9y = 2e^{3t}.$$

undetermined coeff.

$r^2 - 6r + 9 = 0 \Rightarrow r = \frac{6 \pm \sqrt{36-36}}{2} = 3 \Rightarrow \boxed{r_{\pm} = 3}$

$\boxed{y_1(t) = e^{3t}}$, $\boxed{y_2(t) = t e^{3t}}$

5pts

$\boxed{y_p(t) = k t^2 e^{3t}} \leftarrow$ 4pts

$y_p' = (2kt + 3kt^2) \cdot e^{3t}$

$y_p'' = (2k + 6kt) e^{3t} + (2kt + 3kt^2) 3 e^{3t}$

$y_p'' = (2k + 12kt + 9kt^2) e^{3t}$

6pts

$\left[2 + 12t + 9t^2 - 6(2t + 3t^2) + 9t^2\right] k e^{3t} = 2e^{3t}$

$2k = 2 \Rightarrow \boxed{k = 1} \Rightarrow \boxed{y_p = t^2 e^{3t}}$

$\boxed{y(t) = (c_1 + c_2 t + t^2) e^{3t}}$

2pts

15

5. (~~[illegible]~~ points) Find real-valued fundamental solutions to the differential equation

$$y'' + 2y' + 10y = 0.$$

$$r^2 + 2r + 10 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2}$$

$$\boxed{r_\pm = -1 \pm 3i}$$

8 pts

$$\boxed{\begin{aligned} y_1(t) &= e^{-t} \cos(3t) \\ y_2(t) &= e^{-t} \sin(3t) \end{aligned}}$$

7pts

6. (~~17~~ 15 points) Find real-valued fundamental solutions arbitrary close to the point $x_0 = -1$ of the differential equation

$$(x+1)^2\, y'' + 3(x+1)\, y' + 5y = 0, \qquad x \neq -1.$$

Euler eq. $y = |x+1|^r$

$r(r-1) + 3r + 5 = 0$ $\qquad r^2 + 2r + 5 = 0$

$r = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2}$

$\boxed{r_\pm = -1 \pm 2i}$

8pts

$y_1(x) = \dfrac{1}{|x+1|} \cos\left(2 \ln|x+1|\right)$

$y_2(x) = \dfrac{1}{|x+1|} \sin\left(2 \ln|x+1|\right)$

7pts

#	Pts	Score
1	15	
2	~~17~~ 21	
3	17	
4	17	
5	~~17~~ 15	
6	~~17~~ 15	
Σ	100	