Polar coordinates (Sect. 11.3)

- Review: Arc-length of a curve.
- Polar coordinates definition.
- Transformation rules Polar-Cartesian.
- Examples of curves in polar coordinates.

Review: Arc-length of a curve

Definition

A curve on the plane is given in *parametric form* iff it is given by the set of points $\big(x(t), y(t)\big)$, where the parameter $t \in I \subset \mathbb{R}$.

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Remark: If the curve $y = f(x)$ can be described by the parametric functions $\big(x(t), y(t)\big)$, for $t \in I \subset \mathbb{R}$, and if $x'(t) \neq 0$ for $t \in I$, then holds

$$\frac{df}{dx} = \frac{(dy/dt)}{(dx/dt)}.$$

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Remark: The arc-length of a continuously differentiable curve $\big(x(t), y(y)\big)$, for $t \in [a, b]$ is the number

$$L = \int_a^b \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}\, dt.$$

Review: Arc-length of a curve

Remark:

- The formula for the arc-length

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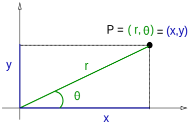

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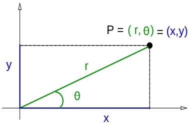

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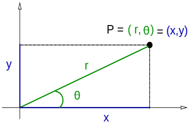

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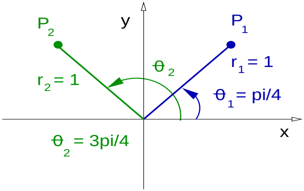

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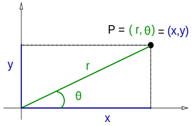

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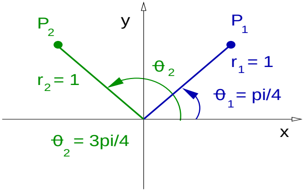

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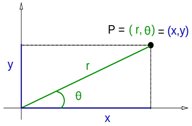

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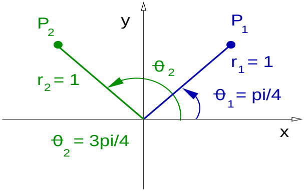

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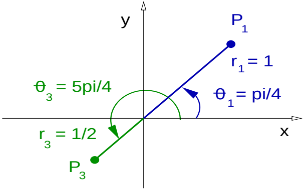

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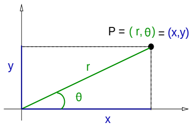

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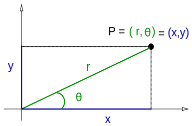

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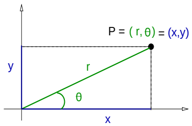

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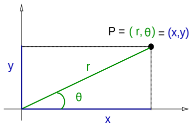

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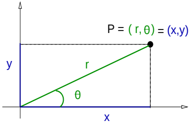

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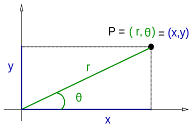

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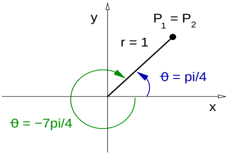

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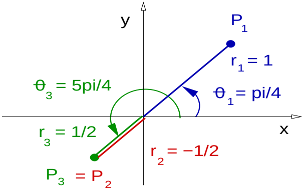

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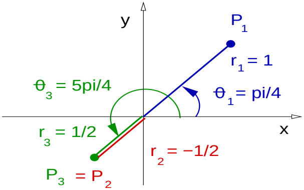
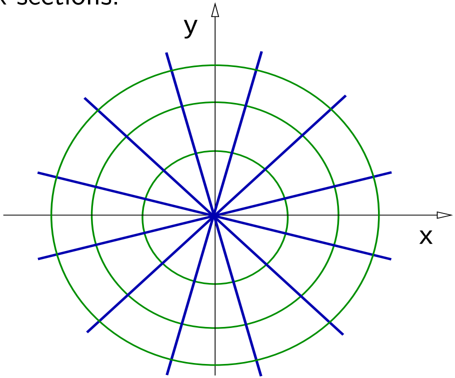


Remark: Polar coordinates are well adapted to describe circular curves and disk sections.



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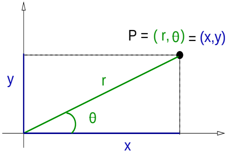

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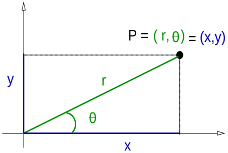

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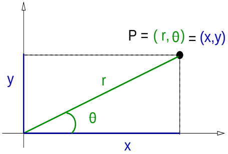

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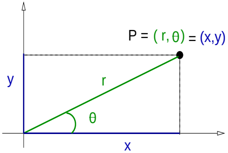

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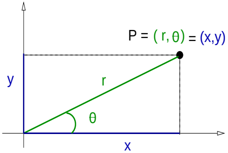

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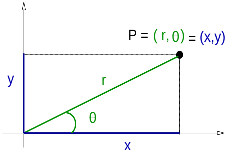

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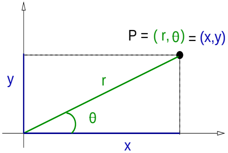

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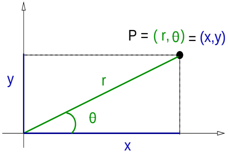

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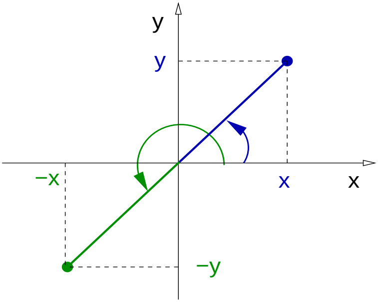

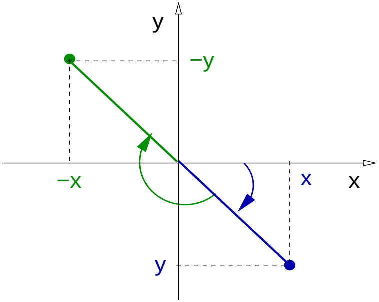

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$$x^2 + y^2 = 3^2, \quad r = \sqrt{x^2 + y^2} \quad \Rightarrow \quad \begin{cases} r = 3, \\ \theta \in [0, 2\pi). \end{cases} \qquad \triangleleft$$

Example

Find the equation in polar coordinates of the line $y = \sqrt{3}\,x$.

Solution: From the transformation laws,

$$\theta = \arctan(y/x) = \arctan\big(\sqrt{3}\big) \quad \Rightarrow \quad \begin{cases} \theta = \pi/3, \\ r \in \mathbb{R}, \end{cases} \qquad \triangleleft$$

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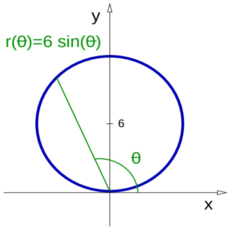

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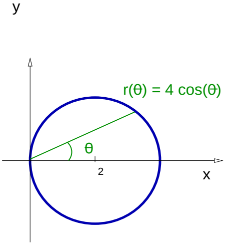

Graphing in polar coordinates (Sect. 11.4)

- Review: Polar coordinates.
- Review: Transforming back to Cartesian.
- Computing the slope of tangent lines.
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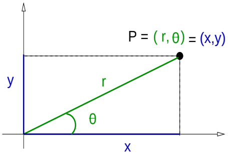
Definition

The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ), with $r > 0$ and $\theta \in [0, 2\pi)$ defined by the picture.



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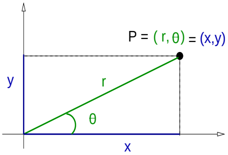

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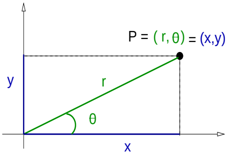

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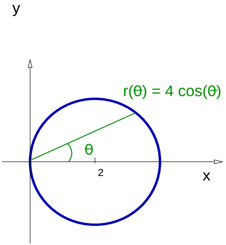

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- **Computing the slope of tangent lines.**
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If the curve passes through the origin, $r(\theta_0) = 0$, then

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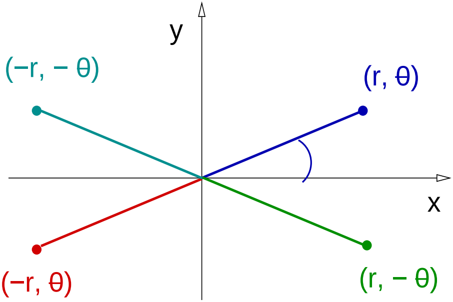

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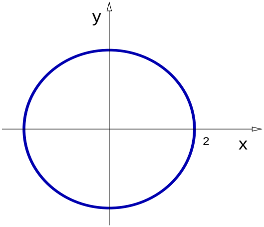

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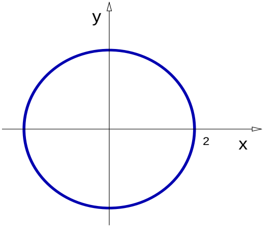

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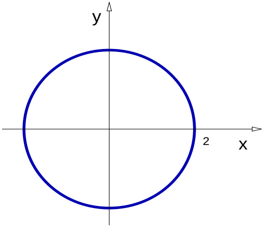

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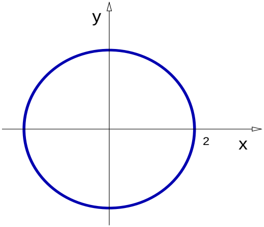

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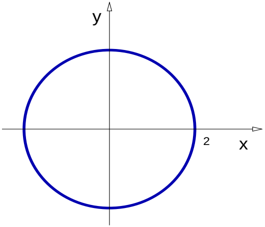
Solution: Back to Cartesian:

Circles in polar coordinates

Remark: Circles centered at the origin are trivial to graph.

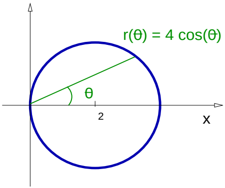


Example

Graph the curve $r = 2, \ \theta \in [0, 2\pi)$.

Remark:

Circles not centered at the origin are more complicated to graph.



Example

Graph the curve

$r = 4\cos(\theta), \ \theta \in [0, 2\pi)$.

Solution: Back to Cartesian:

Circles in polar coordinates

Remark: We now use the graph of the function $r = 4\cos(\theta)$ to graph the curve $r = 4\cos(\theta)$ in the xy-plane.

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Example

Graph the curve $r = 4\cos(\theta), \ \ \theta \in [0, 2\pi)$.

Solution:

Notice that $r(\theta) = r(-\theta)$.
(Reflection about x-axis symmetry.)

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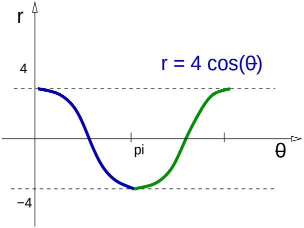

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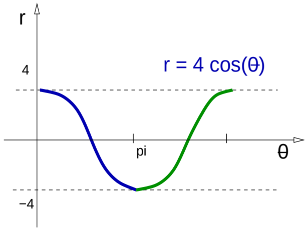

The graph above helps to do the curve on the xy-plane.

Circles in polar coordinates

Remark: We now use the graph of the function $r = 4\cos(\theta)$ to graph the curve $r = 4\cos(\theta)$ in the xy-plane.

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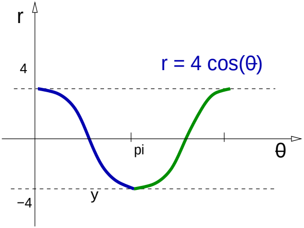
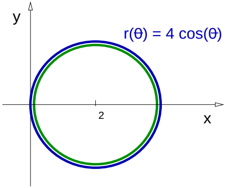


The graph above helps to do the curve on the xy-plane. We actually cover the circle twice!



Graphing in polar coordinates (Sect. 11.4)

- Review: Polar coordinates.
- Review: Transforming back to Cartesian.
- Computing the slope of tangent lines.
- Using symmetry to graph curves.
- Examples:
  - Circles in polar coordinates.
  - **Graphing the Cardiod.**
  - Graphing the Lemniscate.

Graphing the Cardiod

Example

Graph on the xy-plane the curve $r = 1 - \cos(\theta), \;\; \theta \in [0, 2\pi)$.

Graphing the Cardiod

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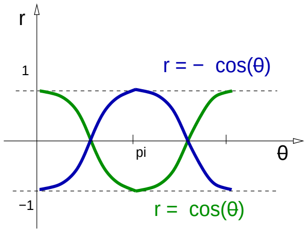

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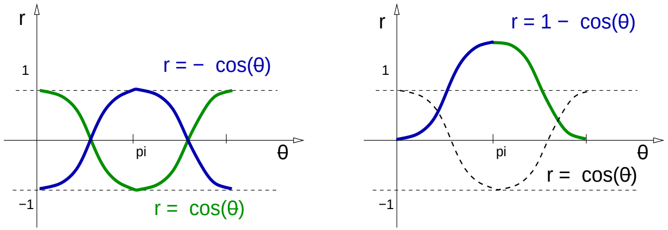

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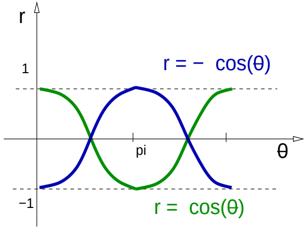
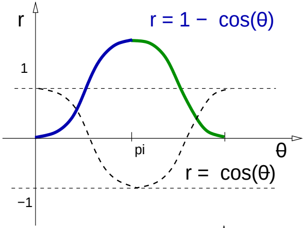
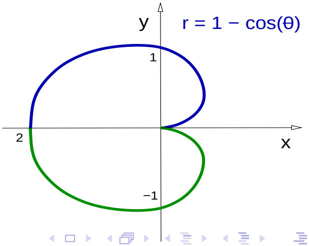




From the previous graph we obtain the curve: on the *xy*-plane:



Graphing the Cardiod

Example

Graph on the xy-plane the curve $r = 1 + \cos(\theta), \ \ \theta \in [0, 2\pi)$.

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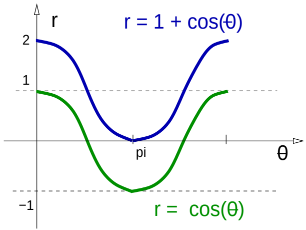

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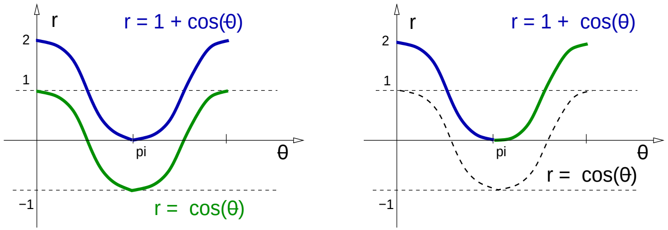

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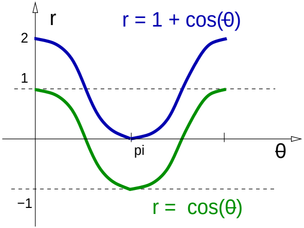

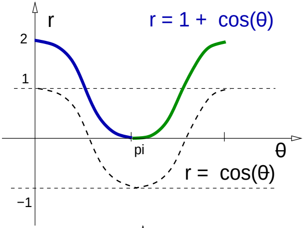

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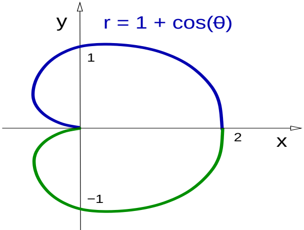

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Graphing the Lemniscate

Example

Graph on the *xy*-plane the curve $r^2 = \sin(2\theta), \;\; \theta \in [0, 2\pi)$.

Graphing the Lemniscate

Example

Graph on the xy-plane the curve $r^2 = \sin(2\theta), \;\; \theta \in [0, 2\pi)$.

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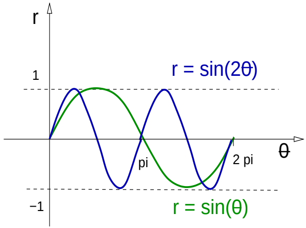

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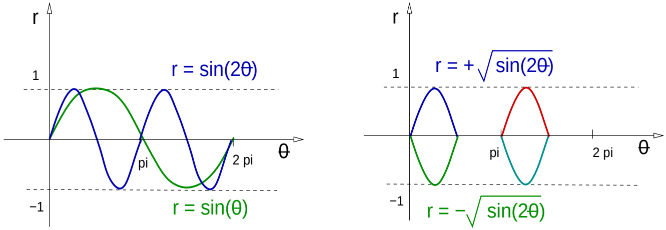

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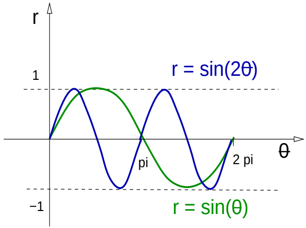

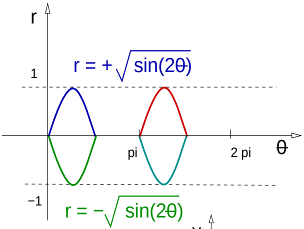

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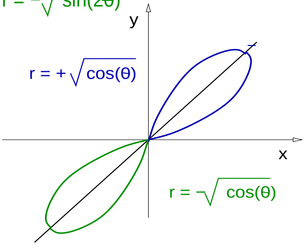


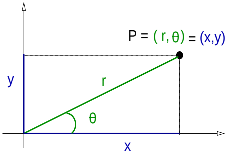
Area of regions in polar coordinates (Sect. 11.5)

- Review: Few curves in polar coordinates.
- Formula for the area or regions in polar coordinates.
- Calculating areas in polar coordinates.

Transformation rules Polar-Cartesian.

Definition

The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ), with $r > 0$ and $\theta \in [0, 2\pi)$ defined by the picture.



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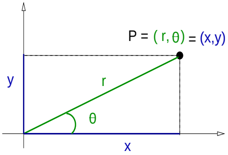

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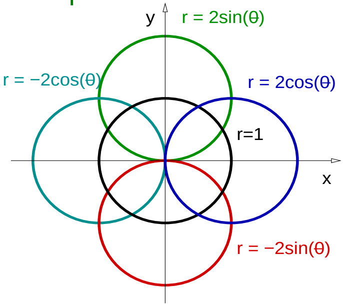

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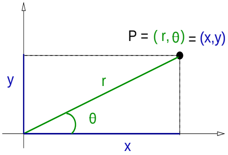

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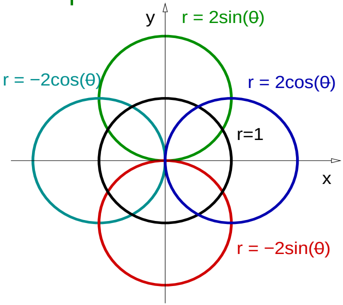

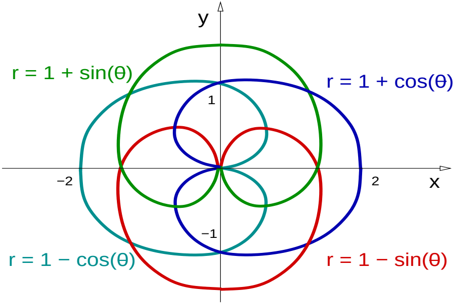

Area of regions in polar coordinates (Sect. 11.5)

- Review: Few curves in polar coordinates.
- **Formula for the area or regions in polar coordinates.**
- Calculating areas in polar coordinates.

Formula for the area or regions in polar coordinates

Theorem

If the functions $r_1, r_2 : [\alpha, \beta] \to \mathbb{R}$ *are continuous and* $0 \leqslant r_1 \leqslant r_2$, *then the area of a region* $D \subset \mathbb{R}^2$ *given by*

$$D = \{(r, \theta) \in \mathbb{R}^2 \;:\; r \in [r_1(\theta), r_2(\theta)],\; \theta \in [\alpha, \beta]\}.$$

is given by the integral

$$A(D) = \int_\alpha^\beta \frac{1}{2} \left(\big[r_2(\theta)\big]^2 - \big[r_1(\theta)\big]^2 \right) d\theta.$$

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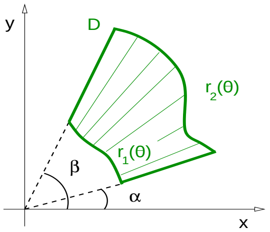

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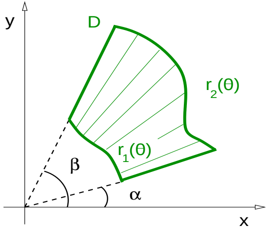

Remark: This result includes the case of $r_1 = 0$, which are fan-shaped regions.

Formula for the area or regions in polar coordinates

Idea of the Proof: Introduce a partition $\theta_k = k\,\Delta\theta$,

Formula for the area or regions in polar coordinates

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Formula for the area or regions in polar coordinates

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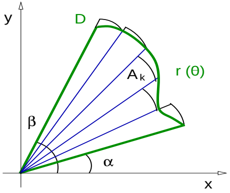

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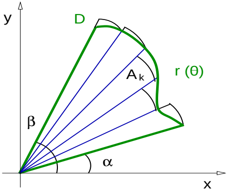

The area of each fan-shaped region on the figure is,

$$A_k = \frac{1}{2}\big[r(\theta_k)\big]^2\,\Delta\theta.$$

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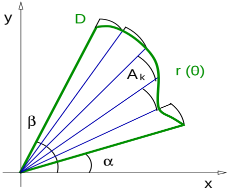

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A Riemann sum that approximates the green region area is

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\big[r(\theta_k)\big]^2\,\Delta\theta.$$

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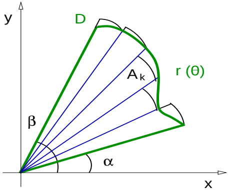

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Refining the partition and taking a limit $n \to \infty$

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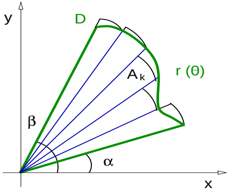

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Refining the partition and taking a limit $n \to \infty$ one can prove that the Riemann sum above converges and the limit is called

$$A(D) = \int_{\alpha}^{\beta} \frac{1}{2}\big[r(\theta)\big]^2\,d\theta. \qquad \square$$

Area of regions in polar coordinates (Sect. 11.5)

- Review: Few curves in polar coordinates.
- Formula for the area or regions in polar coordinates.
- **Calculating areas in polar coordinates.**

Calculating areas in polar coordinates

Example

Find the area inside the circle $r = 1$ and outside the cardiod $r = 1 - \sin(\theta)$.

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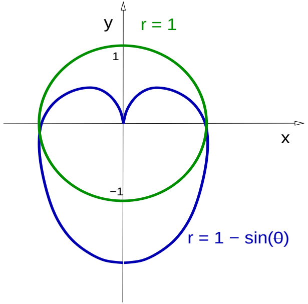

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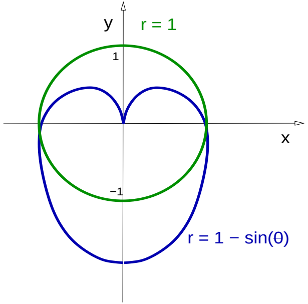

The Theorem implies

$$A = \int_{\alpha}^{\beta} \frac{1}{2} \left(1 - \left[1 - \sin(\theta)\right]^2\right) d\theta.$$

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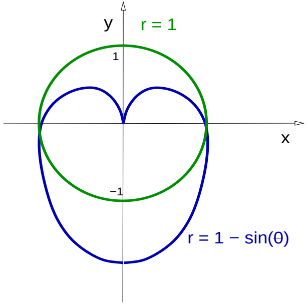

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We need to find α and β.

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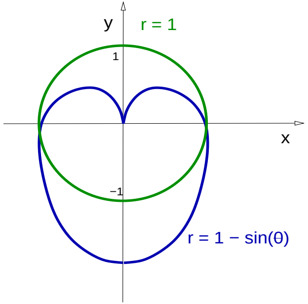

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$$A = \int_{\alpha}^{\beta} \frac{1}{2} \left(1 - \left[1 - \sin(\theta)\right]^2\right) d\theta.$$

We need to find α and β. They are the intersection of the circle and the cardiod:

Calculating areas in polar coordinates

Example

Find the area inside the circle $r = 1$ and outside the cardiod $r = 1 - \sin(\theta)$.

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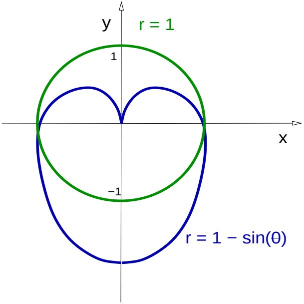

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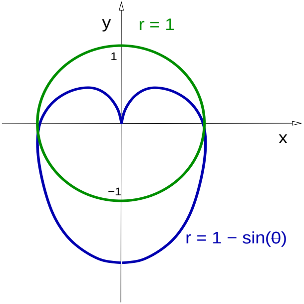

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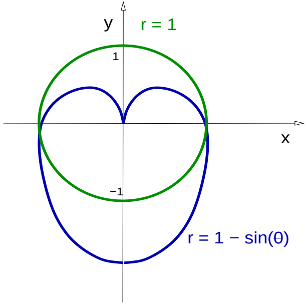

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Calculating areas in polar coordinates

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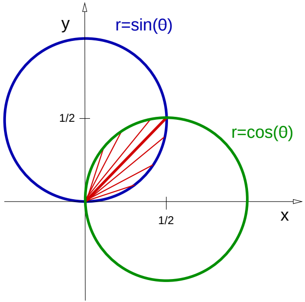

Calculating areas in polar coordinates

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Also works: $A = \int_0^{\pi/4} \frac{1}{2} \sin^2(\theta)\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2} \cos^2(\theta)\, d\theta.$