Binomial functions and Taylor series (Sect. 10.10)

- Review: The Taylor Theorem.
- The binomial function.
- Evaluating non-elementary integrals.
- The Euler identity.
- Taylor series table.

Review: The Taylor Theorem

Recall: If $f : D \to \mathbb{R}$ is infinitely differentiable, and a, $x \in D$, then

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$$R_n(x) = \frac{f^{(n+1)}(c(x))}{(n+1)!}\,(x-a)^{n+1}, \quad \text{with} \quad c \in (a, x).$$

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Furthermore, if $R_n(x) \to 0$ as $n \to \infty$ for every $x \in I \subset D$, then the *Taylor series* centered at $x = a$, $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(x-a)^n$, converges to the function f on the interval I, and $f(x) = T(x)$.

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$$T_3(x) = 1 + m\,x + \frac{m(m-1)}{2!}\,x^2 + \frac{m(m-1)(m-2)}{3!}\,x^3. \quad \lhd$$

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That is, $f_2(x) = T(x)$. $\triangleleft$

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Theorem

The Taylor series for the binomial function $f_m(x) = (1+x)^m$, *with* m *not a positive integer converges for* $|x| < 1$ *and is given by*

$$T(x) = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n,$$

with the binomial coefficients $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$, *and*

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-(n-1))}{n!}.$$

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Proof: The n-derivative of the binomial function is

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Therefore, the series converges for $|x| < 1$. $\square$

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$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots. \qquad \triangleleft$$

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◁

Binomial functions and Taylor series (Sect. 10.10)

- Review: The Taylor Theorem.
- The binomial function.
- **Evaluating non-elementary integrals.**
- The Euler identity.
- Taylor series table.

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- Review: The Taylor Theorem.
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The Euler identity

Remark: The Taylor expansions

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\,, \quad \sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\,,$$

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This and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ suggest the definition:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Binomial functions and Taylor series (Sect. 10.10)

- Review: The Taylor Theorem.
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Taylor series table

Remark: Table of frequently used Taylor series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots, \qquad |x| < 1,$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad |x| < \infty,$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad |x| < \infty,$$

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Parametrizations of curves on a plane (Sect. 11.1)

- Review: Curves on the plane.
- Parametric equations of a curve.
- Examples of curves on the plane.
- The cycloid.

Review: Curves on the plane

Remarks:

- Curves on a plane can be described by the set of points (x, y) solutions of an equation

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- Circle centered at $P = (x_0, y_0)$ radius r:
$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Review: Curves on the plane

Example

- An ellipse centered at $P = (0, 0)$ with radius a and b,

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Parametrizations of curves on a plane (Sect. 11.1)

- Review: Curves on the plane.
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Remark: If the interval I is closed, $I = [a, b]$, then $(x(a), y(a))$ and $(x(b), y(b))$ are called the *initial* and *terminal* points of the curve.

Parametrizations of curves on a plane (Sect. 11.1)

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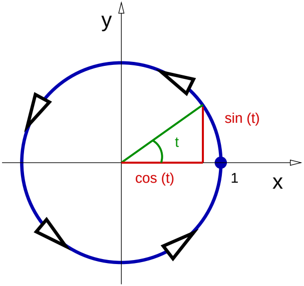

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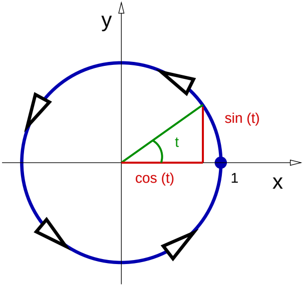

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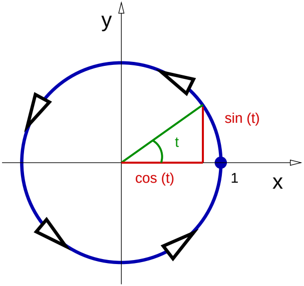

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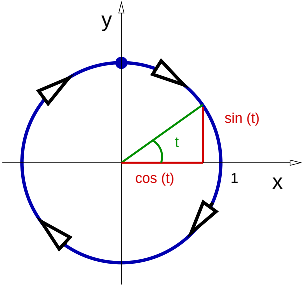

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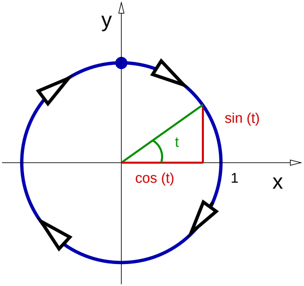

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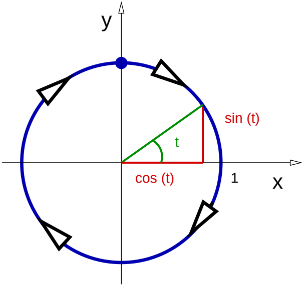

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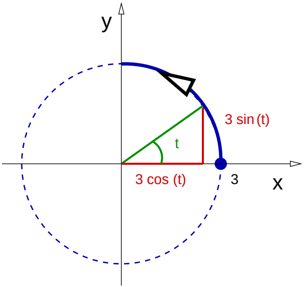

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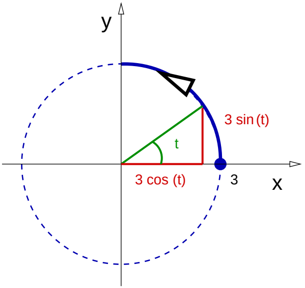

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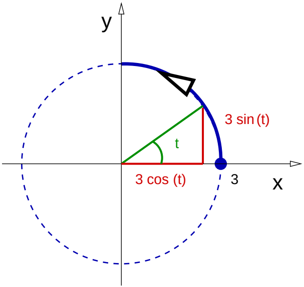

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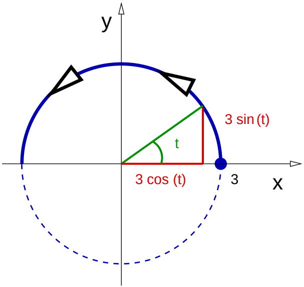

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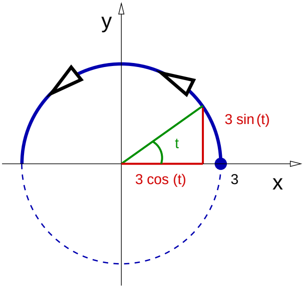

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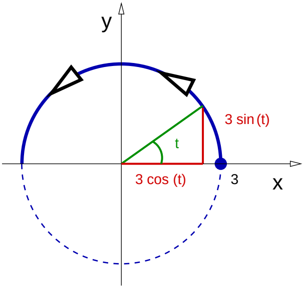

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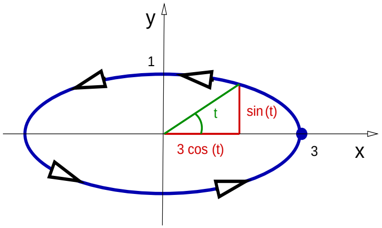

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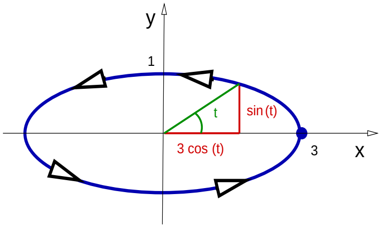

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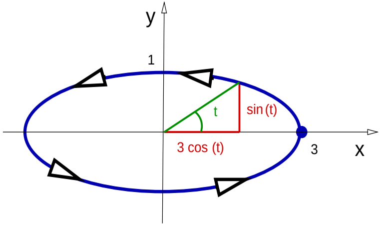

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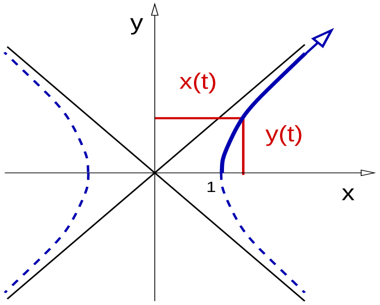

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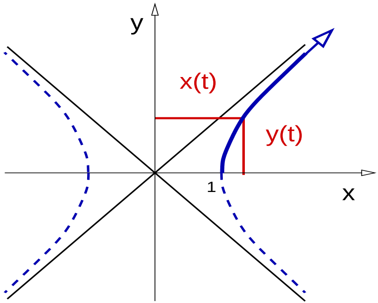

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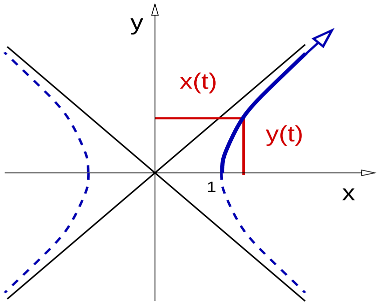

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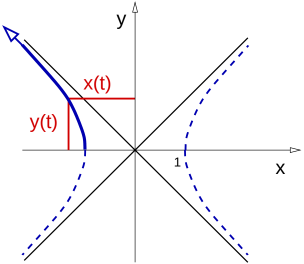

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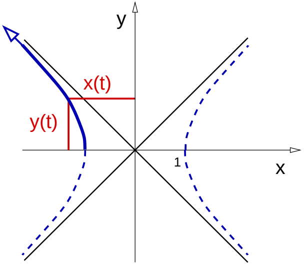

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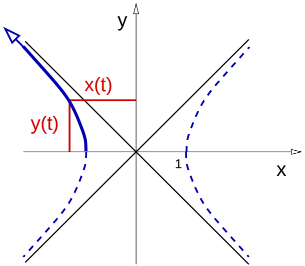

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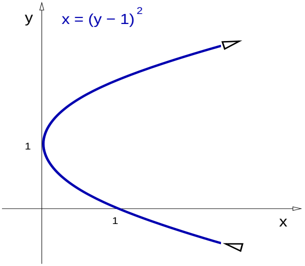

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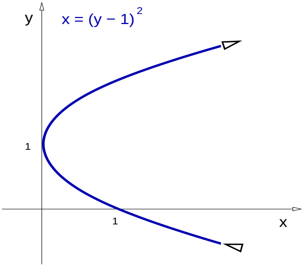

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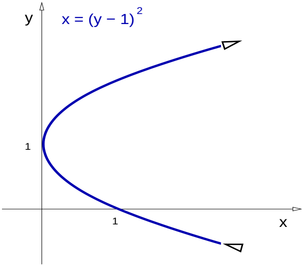

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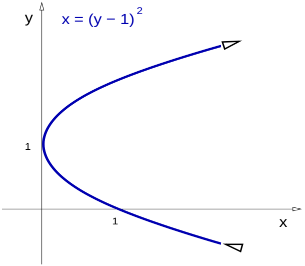

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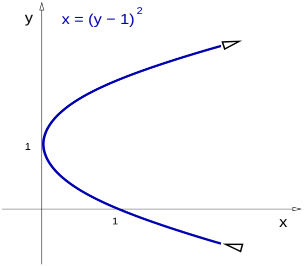

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Parametrizations of curves on a plane (Sect. 11.1)

- Review: Curves on the plane.
- Parametric equations of a curve.
- Examples of curves on the plane.
- **The cycloid.**

The cycloid

Definition

A *cycloid* with parameter $a > 0$ is the curve given by

$$x(t) = a(t - \sin(t)), \quad y(t) = a(1 - \cos(t)), \quad t \in \mathbb{R}.$$

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- The cycloid played an important role in designing precise pendulum clocks, needed for navigation in the 17th century.

Arc-length of a curve on the plane (Sect. 11.2)

- Review: Parametric curves on the plane.
- The slope of tangent lines to curves.
- The arc-length of a curve.
- The arc-length function and differential.

Review: Parametric curves on the plane

Definition

A curve on the plane is given in *parametric form* iff it is given by the set of points $\big(x(t), y(t)\big)$, where the parameter $t \in I \subset \mathbb{R}$.

Review: Parametric curves on the plane

Definition

A curve on the plane is given in *parametric form* iff it is given by the set of points $\big(x(t), y(t)\big)$, where the parameter $t \in I \subset \mathbb{R}$.

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Describe the curve $x(t) = \cosh(t), \;\; y(t) = \sinh(t)$, for $t \in [0, \infty)$.

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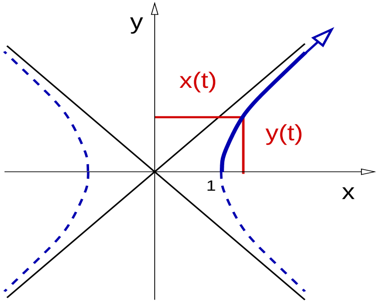

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- This is not the equation of a circle.
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- The cycloid played an important role in designing precise pendulum clocks, needed for navigation in the 17th century.

Arc-length of a curve on the plane (Sect. 11.2)

- Review: Parametric curves on the plane.
- **The slope of tangent lines to curves.**
- The arc-length of a curve.
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The slope of tangent lines to curves

Definition

A curve defined by the parametric function values $\big(x(t), y(t)\big)$, for $t \in I \subset \mathbb{R}$, is *differentiable* iff each function x and y is differentiable on the interval I.

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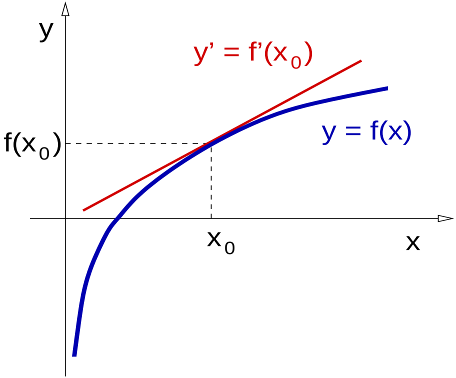
The slope of tangent lines to curves

Remark: The formula $\dfrac{df}{dx} = \dfrac{(dy/dt)}{(dx/dt)}$ provides an alternative way to find the slope of the line tangent to the graph of the function f.



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Remark: In the first quadrant holds $y'(x_0) = \dfrac{-x_0}{\sqrt{1-(x_0)^2}}$. $\triangleleft$

Arc-length of a curve on the plane (Sect. 11.2)

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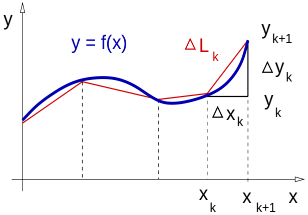
The arc-length of a curve



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The *length* or *arc length* of a curve in the plane or in space is the limit of the polygonal line length, as the polygonal line approximates the original curve.

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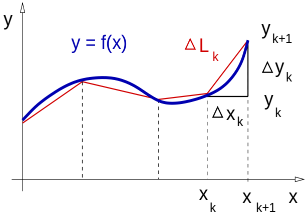


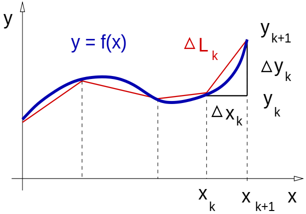
Theorem

The arc-length of a continuously differentiable curve $(x(t), y(y))$, *for* $t \in [a, b]$ *is the number*

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

The arc-length of a curve

Idea of the Proof: The curve length is the limit of the polygonal line length, as the polygonal line approximates the original curve.



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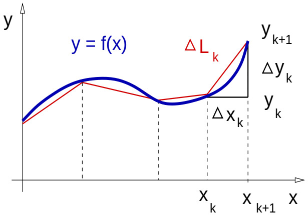

$$L_N = \sum_{n=0}^{N-1} \sqrt{\left(\Delta x_k\right)^2 + \left(\Delta y_k\right)^2} \qquad \{a = t_0, t_1, \cdots, t_{N-1}, t_N = b\},$$

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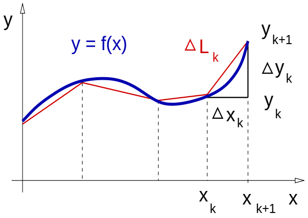

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$$L_N \simeq \sum_{n=0}^{N-1} \sqrt{\left[x'(t_k^*)\right]^2 + \left[y'(t_k^*)\right]^2}\, \Delta t_k,$$

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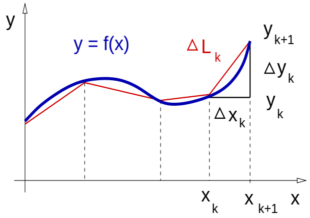

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$$L_N \xrightarrow{N\to\infty} L = \int_a^b \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt.$$

□

The arc-length of a curve

Example

Find the length of the curve $\big(r\cos(t), r\sin(t)\big)$, for $r > 0$ and $t \in [\pi/4, 3\pi/4]$. (Quarter of a circle.)

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Solution: Compute the derivatives

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Hence, $L = \dfrac{\pi}{2}\, r$. (The length of quarter circle of radius r.) $\lhd$

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We obtain $(x')^2 + (y')^2 = t^2 + 1.$

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Solution: The derivative of the parametric curve is

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$$\begin{aligned}(x')^2 + (y')^2 = &\big[t^2\sin^2(t) + \cos^2(t) - 2t\sin(t)\cos(t)\big]\\ &+ \big[t^2\cos^2(t) + \sin^2(t) + 2t\sin(t)\cos(t)\big]\end{aligned}$$

We obtain $(x')^2 + (y')^2 = t^2 + 1$. The curve length is given by

$$L(t_0) = \int_0^{t_0} \sqrt{1 + t^2}\, dt$$

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We conclude that $L(t_0) = \dfrac{t_0}{2}\sqrt{1+t_0^2} + \dfrac{1}{2}\ln\big(t_0 + \sqrt{1+t_0^2}\,\big)$. $\triangleleft$

Arc-length of a curve on the plane (Sect. 11.2)

- Review: Parametric curves on the plane.
- The slope of tangent lines to curves.
- The arc-length of a curve.
- **The arc-length function and differential.**

The arc-length function and differential

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The *arc-length function* of a continuously differentiable curve given by $\big(x(t), y(t)\big)$ for $t \in [t_0, t_1]$ is given by

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(a) The value $L(t)$ of the length function is the length along the curve $\big(x(t), y(t)\big)$ from t_0 to t.

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(b) If the curve is the position of a moving particle as function of time, then the value $L(t)$ is the distance traveled by the particle from the time t_0 to t.

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$$L = \int_0^1 (4t^2 + 6t + 2)\, dt = \Big(\frac{4t^3}{3} + 3\,t^2 + 2t\Big)\Big|_0^1 = \frac{19}{3}. \qquad \triangleleft$$