Power series (Sect. 10.7)

- Power series definition and examples.
- The radius of convergence.
- The ratio test for power series.
- Term by term derivation and integration.

Power series definition and examples

Definition

A power series centered at x_0 is the function $y : D \subset \mathbb{R} \to \mathbb{R}$

$$y(x) = \sum_{n=0}^{\infty} c_n \, (x - x_0)^n, \qquad c_n \in \mathbb{R}.$$

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- An equivalent expression for the power series is

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$$y(x) = c_0 + c_1 \, x + c_2 \, x^2 + c_3 \, x^3 + \cdots .$$
- The domain $D = \{x \in \mathbb{R} \; : \; y(x) \text{ converges.}\}$

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Another examples of power series $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$.

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$$y(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

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The function $y(x) = \dfrac{1}{1-x}$ is defined for $x \in \mathbb{R} - \{1\}$.

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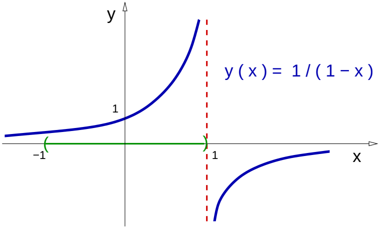

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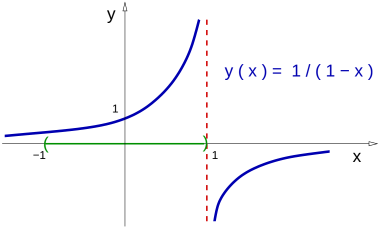

The power series

$$y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

converges only for $|x| < 1$. $\triangleleft$

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The radius of convergence.

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The power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has *radius of convergence* $\rho \geqslant 0$ iff the following conditions hold:

(a) The series converges absolutely for $|x - x_0| < \rho$;

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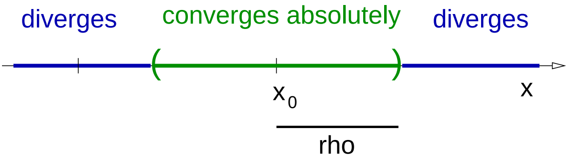

The ratio test for power series

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Determine the radius of convergence and the interval of convergence of the power series $y(x) = \sum_{n=0}^{\infty} x^n$.

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Hence the radius of convergence is $\rho = 1$.

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For the interval of convergence we need to study $y(1)$ and $y(-1)$.

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Both series diverge, since their partial sums do not converge.

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Then the interval of convergence is $I = (-1, 1)$. $\triangleleft$

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Solution: We fix $x \in \mathbb{R}$ and we use the ratio test on the infinite series $\sum_{n=0}^{\infty} \left| \frac{x^n}{n!} \right|$.

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$$\frac{a_{n+1}}{a_n}$$

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as $n \to \infty$, for all $x \in \mathbb{R}$.

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as $n \to \infty$, for all $x \in \mathbb{R}$. The radius of convergence $\rho = \infty$. $\lhd$

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Remark: The interval of convergence is $I = \mathbb{R}$.

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The ratio test for power series

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Determine the radius of convergence of $y(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$.

Solution: Fix $x \in \mathbb{R}$ and use the ratio test on the series $\sum_{n=1}^{\infty} \left| \frac{x^n}{n} \right|$.

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This is a condition on x, since $|x| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$. So $\rho = 1$. $\triangleleft$

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$$x = 1 \;\Rightarrow\; y(1) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -\Big(1 - \frac{1}{2} + \frac{1}{3} - \cdots\Big),$$

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Power series (Sect. 10.7)

- Power series definition and examples.
- The radius of convergence.
- **The ratio test for power series.**
- Term by term derivation and integration.

The ratio test for power series

Theorem (Ratio test for power series)

Given the power series $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, *introduce the number* $L = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|}$. *Then, the following statements hold:*

(1) *The power series converges in the domain* $|x - x_0| L < 1$.

(2) *The power series diverges in the domain* $|x - x_0| L > 1$.

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Proof: $\left| \frac{c_{n+1}(x - x_0)^{n+1}}{c_n (x - x_0)^n} \right| = |x - x_0| \frac{|c_{n+1}|}{|c_n|} \to |x - x_0| L.$ □

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The series converges for $|x| < 8$ and diverges for $|x| > 8$.

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The series converges for $|x| < 8$ and diverges for $|x| > 8$.

The radius of convergence is $\rho = 8$. $\lhd$

The ratio test for power series

Example

Determine the radius of convergence of $y(x) = \sum_{n=0}^{\infty} \frac{x^n}{8^n}$.

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We need to study the series for $x = \pm 8$.

$$x = 8 \quad \Rightarrow \quad y(8) = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 \cdots ,$$

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The interval of convergence is $I = (-8, 8)$. $\triangleleft$

Power series (Sect. 10.7)

- Power series definition and examples.
- The radius of convergence.
- The ratio test for power series.
- **Term by term derivation and integration.**

Term by term derivation and integration

Theorem

If the power series $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ *has radius of convergence* $\rho > 0$*, then function* y *is both differentiable with derivative*

$$y'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{(n-1)},$$

and function y *is integrable with primitive*

$$\int y(x)\, dx = \sum_{n=0}^{\infty} \frac{(x - x_0)^{(n+1)}}{(n+1)} + c,$$

where both expressions above converge on $(x_0 - \rho, x_0 + \rho)$*.*

Taylor Series (Sect. 10.8)

- Review: Power series define functions.
- Functions define power series.
- Taylor series of a function.
- Taylor polynomials of a function.

Review: Power series define functions

Remarks:

- Power series define functions on domains where the series converge.

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Review: Power series define functions

Remarks:

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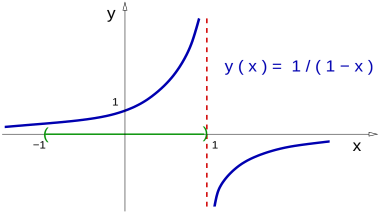

The power series $y(x) = \sum_{n=0}^{\infty} x^n$

Review: Power series define functions

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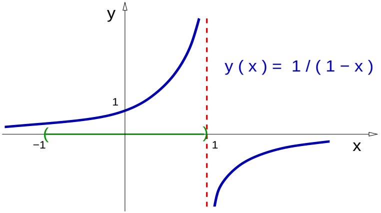

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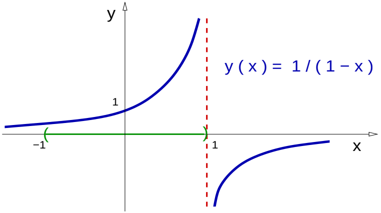

The power series $y(x) = \sum_{n=0}^{\infty} x^n$ converges to the function $f(x) = \dfrac{1}{1-x}$ only on the domain given by $|x| < 1$. $\triangleleft$

Review: Power series define functions

Theorem (Term by term derivation and integration)

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Taylor Series (Sect. 10.8)

- Review: Power series define functions.
- **Functions define power series.**
- Taylor series of a function.
- Taylor polynomials of a function.

Functions define power series

Theorem

If an infinitely differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a power series representation at $a \in D$ with convergence radius $\rho > 0$,

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$$y(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \, (x - a)^n.$$

However, it is not clear whether this series converges at all, and if it does, whether it satisfies that $f(x) = y(x)$ for $x \neq a$.

Functions define power series

Remark: The proof is simple because the assumptions are big.

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Proof: Since the function f has a power series representation,

$$f(x) = c_0 + c_1 (x - a) + c_2 (x - a)^2 + c_3 (x - a)^3 + \cdots$$

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Therefore, $c_n = \dfrac{f^{(n)}(a)}{n!}$. □

Taylor Series (Sect. 10.8)

- Review: Power series define functions.
- Functions define power series.
- **Taylor series of a function.**
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Taylor series of a function

Remark: The Theorem above suggests the following definition.

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Definition

The *Taylor series* centered at $a \in D$ of an infinitely differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is given by

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- The particular case $a = 0$ is called the *Maclaurin series*.

Taylor series of a function

Example

Find the Taylor series of the function $f(x) = \dfrac{1}{x}$ centered at $x = 3$.

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We conclude: $\quad T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n.$ $\triangleleft$

Taylor series of a function

Example

Find the radius of convergence ρ of the Taylor series T centered at $x = a$ of the function $f(x) = \dfrac{1}{x}$.

Taylor series of a function

Example

Find the radius of convergence ρ of the Taylor series T centered at $x = a$ of the function $f(x) = \dfrac{1}{x}$.

Solution: It is simple to see that $T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (x-a)^n$.

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The condition $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} < 1$ implies $\frac{|x-a|}{a} < 1$,

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The condition $\displaystyle\lim_{n\to\infty} \frac{b_{n+1}}{b_n} < 1$ implies $\dfrac{|x-a|}{a} < 1$,

that is $|x-a| < a$. We conclude that $\rho = a$. $\triangleleft$

Taylor Series (Sect. 10.8)

- Review: Power series define functions.
- Functions define power series.
- Taylor series of a function.
- **Taylor polynomials of a function.**

Taylor polynomials of a function

Remark: A truncated Taylor series is called a Taylor polynomial.

Taylor polynomials of a function

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Definition

The *Taylor polynomial* of order n centered at $a \in D$ of an n-differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is given by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

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- The Taylor polynomial of order n centered at $a = 0$ is called the $n+1$ *Maclaurin polynomial*.

Taylor polynomials of a function

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Find the first four Maclaurin polynomials of the function $f(x) = e^{3x}$.

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Solution: Since $f^{(n)}(x) = 3^n\, e^{3x}$, and the polynomials are centered at $a = 0$, the first 4 Maclaurin polynomials are

$$T_0, \quad T_1, \quad T_2, \quad T_3.$$

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$$T_3(x) = 1 + 3x + \frac{3^2}{2}\,x^2 + \frac{3^3}{6}\,x^3. \qquad \triangleleft$$

Taylor polynomials of a function

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Find the first seven Maclaurin polynomials of $f(x) = \cos(x)$.

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Solution: $f(x) = \cos(x), \;\; f'(x) = -\sin(x),$

Taylor polynomials of a function

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Solution: $f(x) = \cos(x), \;\; f'(x) = -\sin(x), \;\; f''(x) = -\cos(x),$

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Solution: $f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$,

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$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1.$$

Since $T_n(x) = f(0) + f'(0)\,x + \dfrac{f''(0)}{2!}\,x^2 + \cdots + \dfrac{f(n)(0)}{n!}\,x^n$,

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Since $T_n(x) = f(0) + f'(0)\,x + \dfrac{f''(0)}{2!}\,x^2 + \cdots + \dfrac{f(n)(0)}{n!}\,x^n$, then

$$T_0(x) = 1 = T_1(x),$$

Taylor polynomials of a function

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Find the first seven Maclaurin polynomials of $f(x) = \cos(x)$.

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$$T_0(x) = 1 = T_1(x), \quad T_2(x) = 1 - \frac{x^2}{2}$$

Taylor polynomials of a function

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Find the first seven Maclaurin polynomials of $f(x) = \cos(x)$.

Solution: $f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and then the derivatives repeat,

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1.$$

Since $T_n(x) = f(0) + f'(0)\,x + \dfrac{f''(0)}{2!}\,x^2 + \cdots + \dfrac{f(n)(0)}{n!}\,x^n$, then

$$T_0(x) = 1 = T_1(x), \quad T_2(x) = 1 - \frac{x^2}{2} = T_3(x),$$

Taylor polynomials of a function

Example

Find the first seven Maclaurin polynomials of $f(x) = \cos(x)$.

Solution: $f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and then the derivatives repeat,

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1.$$

Since $T_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f(n)(0)}{n!}\,x^n$, then

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$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

Taylor polynomials of a function

Example

Find the first seven Maclaurin polynomials of $f(x) = \cos(x)$.

Solution: $f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and then the derivatives repeat,

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$$T_0(x) = 1 = T_1(x), \quad T_2(x) = 1 - \frac{x^2}{2} = T_3(x),$$

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} = T_5(x),$$

Taylor polynomials of a function

Example

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Solution: $f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and then the derivatives repeat,

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1.$$

Since $T_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f(n)(0)}{n!}\,x^n$, then

$$T_0(x) = 1 = T_1(x), \quad T_2(x) = 1 - \frac{x^2}{2} = T_3(x),$$

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} = T_5(x), \quad T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}. \quad \triangleleft$$

Taylor polynomials of a function

Remark: The Taylor polynomial order n centered at any point $x = a$ of a polynomial degree n, say P_n, is the same polynomial.

Taylor polynomials of a function

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Find the T_2 centered at $x = a$ of $f(x) = x^2 + x + 1$.

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Example

Find the T_2 centered at $x = a$ of $f(x) = x^2 + x + 1$.

Solution: Since $f'(x) = 2x + 1$

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Example

Find the T_2 centered at $x = a$ of $f(x) = x^2 + x + 1$.

Solution: Since $f'(x) = 2x + 1$ and $f''(x) = 2$,

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Solution: Since $f'(x) = 2x + 1$ and $f''(x) = 2$, then

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Since $T_2(x) = f(a) + f'(a)\,(x - a) + \dfrac{f''(a)}{2!}\,(x - a)^2$,

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Taylor polynomials of a function

Remark: The Taylor polynomial order n centered at any point $x = a$ of a polynomial degree n, say P_n, is the same polynomial. That is, $T_n(x) = P_n(x)$.

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Hence $T_2(x) = 1 + x + x^2$. ◁

Convergence of Taylor Series (Sect. 10.9)

- Review: Taylor series and polynomials.
- The Taylor Theorem.
- Using the Taylor series.
- Estimating the remainder.

Review: Taylor series and polynomials

Definition

The *Taylor series* and *Taylor polynomial* order n centered at $a \in D$ of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ are given by

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- The particular case $a = 0$ is called the *Maclaurin series* and the $n+1$ *Maclaurin polynomial*, respectively.

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Theorem (Taylor's Theorem)

If $f : [a, b] \to \mathbb{R}$ is $(n + 1)$-times continuously differentiable, then there there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)\,(b - a) + \frac{f''(a)}{2}\,(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}\,(b - a)^{n+1}.$$

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Remark: We can use the Taylor polynomial to write that

$$f(x) = T_n(x) + R_n(x).$$

The Taylor Theorem

Corollary

Let $f : D \to \mathbb{R}$ be infinitely differentiable with Taylor polynomials T_n and remainders R_n, that is, for $n \geqslant 1$ holds

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Remark: Without knowing $c(x)$ it is often possible to estimate

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Show that the Taylor series of $f(x) = e^x$ centered at $a = 0$ converges on $\mathbb{R}$.

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Solution: The straightforward way is to compute the derivatives $f^{(n)}(x)$. A simpler ways is to realize that

$$f(x) = 3x^2 \left[\frac{1}{(1-x)^3}\right] = 3x^2 \, \frac{1}{2} \left[\frac{1}{(1-x)^2}\right]' = \frac{3}{2} \, x^2 \left[\frac{1}{(1-x)}\right]''$$

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We conclude: $f(x) = \dfrac{3}{2}\big[2x^2 + (3)(2)x^3 + (4)(3)x^4 + \cdots\big].$ $\lhd$

Using the Taylor series

Example

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Solution: If $y(x) = \cos(x)$, $\;y'(x) = -\sin(x)$,

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Substitute x by $2\sqrt{x}$ above, $\cos\big(2\sqrt{x}\big) = \sum_{n=0}^{\infty} (-1)^n \frac{(2\sqrt{x})^{2n}}{(2n)!}$.

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Substitute x by $2\sqrt{x}$ above, $\cos\big(2\sqrt{x}\big) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(2\sqrt{x})^{2n}}{(2n)!}$.

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◁

Convergence of Taylor Series (Sect. 10.9)

- Review: Taylor series and polynomials.
- The Taylor Theorem.
- Using the Taylor series.
- **Estimating the remainder.**

Estimating the remainder

Theorem

Let $f : D \to \mathbb{R}$ be infinitely differentiable with Taylor polynomials T_n and remainders R_n centered at $a \in D$, that is, for $n \geqslant 1$ holds

$$f(x) = T_n(x) + R_n(x).$$

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If $|f^{(n+1)}(y)| \leqslant M$ for all y such that $|y - a| \leqslant |x - a|$, then

$$|R_n(x)| \leqslant M \frac{|x - a|^{n+1}}{(n+1)!}.$$

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Furthermore, if the inequality above holds for every $n \geqslant 1$, then the Taylor series $T(x)$ converges to $f(x)$.

Estimating the remainder

Example

Estimate the maximum error made in approximating $f(x) = e^x$ by $T_2(x) = 1 + x + \frac{x^2}{2}$ over the interval $[-2, 2]$.

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Therefore, the smallest bound for R_2 in $[-2, 2]$ is

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$$|f^{(3)}(y)| \leqslant e^2 = M \quad \text{for} \quad |y - 0| \leqslant |2 - 0| = 2.$$

Therefore, the smallest bound for R_2 in $[-2, 2]$ is

$$|R_2(x)| \leqslant e^2 \frac{|x|^3}{3!} \leqslant e^2 \frac{2}{6}$$

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Therefore, the smallest bound for R_2 in $[-2, 2]$ is

$$|R_2(x)| \leqslant e^2 \frac{|x|^3}{3!} \leqslant e^2 \frac{2}{6} \quad \Rightarrow \quad |R_2(x)| \leqslant \frac{e^2}{3}. \qquad \triangleleft$$