# Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- Direct comparison test for series.
- ▶ Review: Limit comparison test for integrals.

- Limit comparison test for series.
- Few examples.

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#### Review: Direct comparison test for integrals.

- Direct comparison test for series.
- Review: Limit comparison test for integrals.
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- Few examples.

Theorem (Direct comparison test)  
If 
$$0 \leq \int_{a}^{\infty} f(x) dx \leq \int_{a}^{\infty} g(x) dx$$
, then:  
(a)  $\int_{a}^{\infty} g(x) dx$  converges  $\Rightarrow \int_{a}^{\infty} f(x) dx$  converges;  
(b)  $\int_{a}^{\infty} f(x) dx$  diverges  $\Rightarrow \int_{a}^{\infty} g(x) dx$  diverges.

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$$\int_0^\infty e^{-x^2} \, dx \, \text{ converges,}$$

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$$\int_0^\infty e^{-x^2} dx \text{ converges, since } \int_0^\infty e^{-x^2} dx \leqslant \int_0^\infty e^{-x} dx.$$

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$$\int_{0}^{\infty} e^{-x^{2}} dx \text{ converges, since } \int_{0}^{\infty} e^{-x^{2}} dx \leqslant \int_{0}^{\infty} e^{-x} dx.$$
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# Comparison tests (Sect. 10.4)

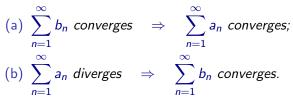
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#### Theorem

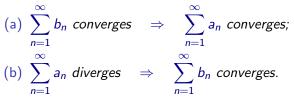
If the sequences satisfy  $0 \leq a_n \leq b_n$  for all  $n \geq N$ , then

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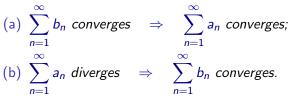
### Example

Determine whether the the series  $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$  converges or not.

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#### Example

Determine whether the the series  $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$  converges or not. Solution: Since  $\frac{n+2}{n^2-n} > \frac{n}{n^2-n}$ 

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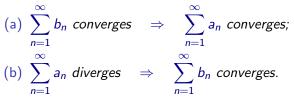
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Determine whether the the series  $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$  converges or not. Solution: Since  $\frac{n+2}{n^2-n} > \frac{n}{n^2-n} = \frac{1}{n-1}$ 

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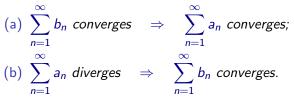
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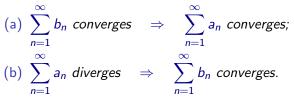


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Determine whether the the series  $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$  converges or not. Solution: Since  $\frac{n+2}{n^2-n} > \frac{n}{n^2-n} = \frac{1}{n-1} > \frac{1}{n}$ , we conclude that:

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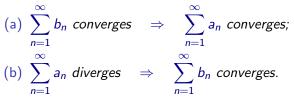


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Determine whether the the series  $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$  converges or not. Solution: Since  $\frac{n+2}{n^2-n} > \frac{n}{n^2-n} = \frac{1}{n-1} > \frac{1}{n}$ , we conclude that:  $\sum_{n=2}^{\infty} \frac{1}{n} < \sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ . Therefore,  $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$  diverges.

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Determine whether the the series  $\sum_{n=1}^{\infty} \frac{1}{n 3^n}$  converges or not.

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Solution: For  $1 \leq n$  holds,

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$$\sum_{n=1}^{\infty} \frac{1}{n \, 3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n}$$

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$$\sum_{n=1}^{\infty} \frac{1}{n \, 3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

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$$\sum_{n=1}^{\infty} \frac{1}{n \, 3^n} \leqslant \frac{1}{\left(1-\frac{1}{3}\right)} - 1$$

#### Example

Determine whether the the series  $\sum_{n=1}^{\infty} \frac{1}{n 3^n}$  converges or not.

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 $\sum_{n=1}^{\infty} \frac{1}{n \, 3^n} \leqslant \frac{1}{\left(1 - \frac{1}{3}\right)} - 1 = \frac{1}{\left(\frac{3-1}{3}\right)} - 1$ 

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We conclude that  $\sum_{n=1}^{\infty} \frac{1}{n \, 3^n}$  converges.

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- Direct comparison test for series.
- ► Review: Limit comparison test for integrals.

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- Limit comparison test for series.
- Few examples.

Theorem (Limit comparison test) If positive functions  $f, g : [a, \infty) \to \mathbb{R}$  are continuous and  $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad with \quad 0 < L < \infty,$ then either both  $\int_{a}^{\infty} f(x) dx, \quad \int_{a}^{\infty} g(x) dx$ , converge or diverge.

Theorem (Limit comparison test) If positive functions  $f, g : [a, \infty) \to \mathbb{R}$  are continuous and  $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad \text{with} \quad 0 < L < \infty,$ 

then either both 
$$\int_{a}^{\infty} f(x) dx$$
,  $\int_{a}^{\infty} g(x) dx$ , converge or diverge.

Remark: If the integrals converge, their values may not agree.

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$$\lim_{x\to\infty}\frac{r(x)}{g(x)}=L,\quad \text{with}\quad 0< L<\infty,$$

then either both  $\int_{a}^{\infty} f(x) dx$ ,  $\int_{a}^{\infty} g(x) dx$ , converge or diverge.

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Remark: If the integrals converge, their values may not agree.

#### Example

 $\int_1^\infty \frac{dx}{\sqrt{x^6+1}} \text{ converges }$ 

Theorem (Limit comparison test) If positive functions  $f, g : [a, \infty) \to \mathbb{R}$  are continuous and

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=L,\quad \text{with}\quad 0< L<\infty,$$

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$$\int_{1}^{\infty} \frac{dx}{\sqrt{x^{6}+1}} \text{ converges because } \int_{1}^{\infty} \frac{dx}{x^{3}} \text{ converges.}$$

Theorem (Limit comparison test) If positive functions  $f, g : [a, \infty) \to \mathbb{R}$  are continuous and  $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad \text{with} \quad 0 < L < \infty,$ 

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Remark: If the integrals converge, their values may **not** agree.

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$$\int_{1}^{\infty} \frac{dx}{\sqrt{x^{6} + 1}} \text{ converges because } \int_{1}^{\infty} \frac{dx}{x^{3}} \text{ converges.}$$
$$\int_{1}^{\infty} \frac{dx}{\sqrt{x + \sin(x)}} \text{ diverges}$$

Theorem (Limit comparison test) If positive functions  $f, g : [a, \infty) \to \mathbb{R}$  are continuous and  $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad \text{with} \quad 0 < L < \infty,$ 

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$$\int_{1}^{\infty} \frac{dx}{\sqrt{x^{6}+1}} \text{ converges because } \int_{1}^{\infty} \frac{dx}{x^{3}} \text{ converges.}$$
$$\int_{1}^{\infty} \frac{dx}{\sqrt{x+\sin(x)}} \text{ diverges because } \int_{1}^{\infty} \frac{dx}{x^{1/2}} \text{ diverges.}$$

# Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- Direct comparison test for series.
- Review: Limit comparison test for integrals.

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- Limit comparison test for series.
- Few examples.

Theorem (Limit comparison test) Assume that  $0 < a_n$ , and  $0 < b_n$  for  $N \leq n$ .

(a) If  $\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$ , then the infinite series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  both converge or both diverge.

(b) If 
$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0$$
, and  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.  
(c) If  $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ , and  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  diverges.

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Remark: If the series converge, their values may **not** agree.

Example

Determine whether the series

es 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$$
 converges or not.

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Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} rac{\sqrt{n}}{4n^2+7}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

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 $\frac{\sqrt{n}}{(4n^2+7)}$ 

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \, \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$

Example

Determine whether the series 
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For *n* large  $a_n = \frac{\sqrt{n}}{(4n^2+7)}$  behaves like

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For *n* large  $a_n = \frac{\sqrt{n}}{(4n^2+7)}$  behaves like  $b_n = \frac{1}{4n^{3/2}}$ .

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} rac{\sqrt{n}}{4n^2+7}$$
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$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$
  
For *n* large  $a_n = \frac{\sqrt{n}}{(4n^2+7)}$  behaves like  $b_n = \frac{1}{4n^{3/2}}$ .  
We choose  $b_n = \frac{1}{4n^{3/2}}$  to do the limit comparison test.

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} rac{\sqrt{n}}{4n^2+7}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$
  
For *n* large  $a_n = \frac{\sqrt{n}}{(4n^2+7)}$  behaves like  $b_n = \frac{1}{4n^{3/2}}$ .  
We choose  $b_n = \frac{1}{4n^{3/2}}$  to do the limit comparison test.  
$$\lim_{n \to \infty} \frac{a_n}{b_n} = \left(\frac{\sqrt{n}}{(4n^2+7)}\right) 4n^{3/2}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} rac{\sqrt{n}}{4n^2+7}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$
  
For *n* large  $a_n = \frac{\sqrt{n}}{(4n^2+7)}$  behaves like  $b_n = \frac{1}{4n^{3/2}}$ .  
We choose  $b_n = \frac{1}{4n^{3/2}}$  to do the limit comparison test.  
$$\lim_{n \to \infty} \frac{a_n}{b_n} = \left(\frac{\sqrt{n}}{(4n^2+7)}\right) 4n^{3/2} = \lim_{n \to \infty} \frac{4n^2}{(4n^2+7)}$$

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Example

Determine whether the series 
$$\sum_{n=1}^{\infty} rac{\sqrt{n}}{4n^2+7}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4 + \left(\frac{7}{n^2}\right)}$$
  
For *n* large  $a_n = \frac{\sqrt{n}}{(4n^2+7)}$  behaves like  $b_n = \frac{1}{4n^{3/2}}$ .  
We choose  $b_n = \frac{1}{4n^{3/2}}$  to do the limit comparison test.  
$$\lim_{n \to \infty} \frac{a_n}{b_n} = \left(\frac{\sqrt{n}}{(4n^2+7)}\right) 4n^{3/2} = \lim_{n \to \infty} \frac{4n^2}{(4n^2+7)} = 1.$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$$
 converges or not.

Solution: 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$$
 and  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  both converge or diverge.

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$$
 converges or not.  
Solution:  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$  and  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  both converge or diverge.

However, 
$$\sum_{n=1}^{\infty} \frac{1}{4 n^{3/2}}$$
 converges  $\Leftrightarrow \int_{1}^{\infty} \frac{dx}{4 x^{3/2}}$  converges.

Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$  converges or not. Solution:  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$  and  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  both converge or diverge. However,  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  converges  $\Leftrightarrow \int_{1}^{\infty} \frac{dx}{4x^{3/2}}$  converges. But:  $\int_{1}^{\infty} \frac{dx}{4x^{3/2}}$ 

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Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$$
 converges or not.  
Solution:  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$  and  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  both converge or diverge.  
However,  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  converges  $\Leftrightarrow \int_{1}^{\infty} \frac{dx}{4x^{3/2}}$  converges.  
But:  $\int_{1}^{\infty} \frac{dx}{4x^{3/2}} = \frac{1}{4}(-2)x^{-1/2}\Big|_{1}^{\infty}$ 

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$$
 converges or not.

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Solution: 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$$
 and  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  both converge or diverge.  
However,  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  converges  $\Leftrightarrow \int_{1}^{\infty} \frac{dx}{4x^{3/2}}$  converges.  
But:  $\int_{1}^{\infty} \frac{dx}{4x^{3/2}} = \frac{1}{4}(-2)x^{-1/2}\Big|_{1}^{\infty} = \frac{1}{2}.$ 

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$$
 converges or not.  
Solution:  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$  and  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  both converge or diverge.

However,  $\sum_{n=1}^{\infty} 4n^2 + 7$  and  $\sum_{n=1}^{\infty} 4n^{3/2}$  beth converge of divergent However,  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  converges  $\Leftrightarrow \int_{1}^{\infty} \frac{dx}{4x^{3/2}}$  converges. But:  $\int_{1}^{\infty} \frac{dx}{4x^{3/2}} = \frac{1}{4}(-2)x^{-1/2}\Big|_{1}^{\infty} = \frac{1}{2}$ . Then, the integral test says that  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  converges.

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$  converges or not. Solution:  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$  and  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  both converge or diverge. However,  $\sum_{n=1}^{\infty} \frac{1}{4n^{3/2}}$  converges  $\Leftrightarrow \int_{1}^{\infty} \frac{dx}{4x^{3/2}}$  converges. But:  $\int_{-\infty}^{\infty} \frac{dx}{4x^{3/2}} = \frac{1}{4} (-2) x^{-1/2} \Big|_{1}^{\infty} = \frac{1}{2}.$ Then, the integral test says that  $\sum_{n=1}^\infty \frac{1}{4\,n^{3/2}}\,$  converges. The limit test for series says that  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$  converges.

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$  converges or not.

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Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\lim_{n\to\infty}\frac{3^{2n}}{2^n+n}=\lim_{n\to\infty}\frac{3^{2n}}{2^n}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

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$$\lim_{n \to \infty} \frac{3^{2n}}{2^n + n} = \lim_{n \to \infty} \frac{3^{2n}}{2^n} \text{ and } \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\lim_{n \to \infty} \frac{3^{2n}}{2^n + n} = \lim_{n \to \infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\lim_{n \to \infty} \frac{3^{2n}}{2^n + n} = \lim_{n \to \infty} \frac{3^{2n}}{2^n} \text{ and } \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
  
For *n* large  $a_n = \frac{3^{2n}}{(2^n + n)}$  behaves like

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\lim_{n \to \infty} \frac{3^{2n}}{2^n + n} = \lim_{n \to \infty} \frac{3^{2n}}{2^n} \text{ and } \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
  
For *n* large  $a_n = \frac{3^{2n}}{(2^n + n)}$  behaves like  $b_n = \left(\frac{3}{\sqrt{2}}\right)^{2n}$ .

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\lim_{n \to \infty} \frac{3^{2n}}{2^n + n} = \lim_{n \to \infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
  
For *n* large  $a_n = \frac{3^{2n}}{(2^n + n)}$  behaves like  $b_n = \left(\frac{3}{\sqrt{2}}\right)^{2n}$ .  
We choose  $b_n = \left(\frac{3}{\sqrt{2}}\right)^{2n}$  to do the limit comparison test,

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\lim_{n \to \infty} \frac{3^{2n}}{2^n + n} = \lim_{n \to \infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
  
For *n* large  $a_n = \frac{3^{2n}}{(2^n + n)}$  behaves like  $b_n = \left(\frac{3}{\sqrt{2}}\right)^{2n}$ .  
We choose  $b_n = \left(\frac{3}{\sqrt{2}}\right)^{2n}$  to do the limit comparison test, hence  
$$\lim_{n \to \infty} \frac{a_n}{2^n} = 1$$

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 $\lim_{n\to\infty} \frac{1}{b_n} =$ 

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\lim_{n \to \infty} \frac{3^{2n}}{2^n + n} = \lim_{n \to \infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
  
For *n* large  $a_n = \frac{3^{2n}}{(2^n + n)}$  behaves like  $b_n = \left(\frac{3}{\sqrt{2}}\right)^{2n}$ .  
We choose  $b_n = \left(\frac{3}{\sqrt{2}}\right)^{2n}$  to do the limit comparison test, hence

$$\lim_{n\to\infty}\frac{a_n}{b_n}=1 \text{ and both } \sum_{n=1}^{\infty}a_n, \sum_{n=1}^{\infty}b_n \text{ converge or diverge.}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
 converges or not.  
Solution: Both  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ , and  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  converge or diverge.

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$  converges or not. Solution: Both  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ , and  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  converge or diverge. Since  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  is a geometric series

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$  converges or not. Solution: Both  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ , and  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  converge or diverge.

Since 
$$\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
 is a geometric series with ratio  $r = \frac{3}{\sqrt{2}}$ 

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$  converges or not.

Solution: Both 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
, and  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  converge or diverge.

Since 
$$\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
 is a geometric series with ratio  $r = \frac{3}{\sqrt{2}} > 1$ ,

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$  converges or not.

Solution: Both 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
, and  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  converge or diverge.

Since 
$$\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
 is a geometric series with ratio  $r = \frac{3}{\sqrt{2}} > 1$ ,

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the series  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ 

Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$  converges or not.

Solution: Both 
$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$$
, and  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  converge or diverge.

Since 
$$\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$$
 is a geometric series with ratio  $r = \frac{3}{\sqrt{2}} > 1$ ,

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the series  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  diverges.

## Limit comparison test for series

Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$  converges or not. Solution: Both  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ , and  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  converge or diverge. Since  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  is a geometric series with ratio  $r = \frac{3}{\sqrt{2}} > 1$ , the series  $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$  diverges. We conclude that  $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$  diverges.  $\triangleleft$ 

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## Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- Direct comparison test for series.
- Review: Limit comparison test for integrals.

- Limit comparison test for series.
- ► Few examples.

#### Example

(1) 
$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}.$$

#### Example

(1) 
$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$$
. DGC

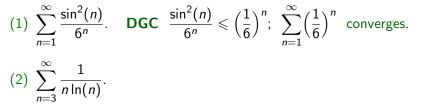
#### Example

(1) 
$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$$
. DGC  $\frac{\sin^2(n)}{6^n} \leq \left(\frac{1}{6}\right)^n$ ;

#### Example

(1) 
$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$$
. DGC  $\frac{\sin^2(n)}{6^n} \leq \left(\frac{1}{6}\right)^n$ ;  $\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$  converges.

#### Example



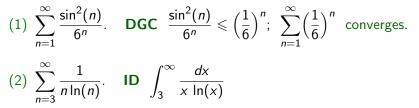
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#### Example

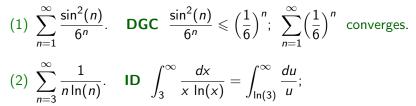
(1) 
$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$$
. DGC  $\frac{\sin^2(n)}{6^n} \leq \left(\frac{1}{6}\right)^n$ ;  $\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$  converges.

(2) 
$$\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$$
. ID

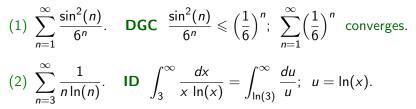
#### Example



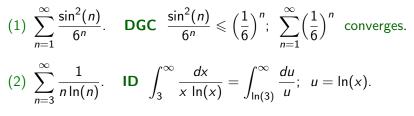
#### Example



#### Example

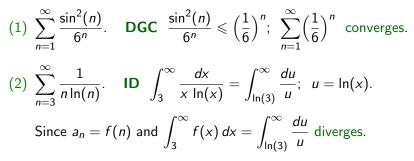


#### Example

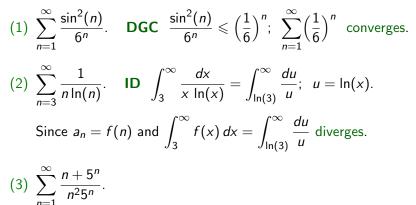


Since 
$$a_n = f(n)$$

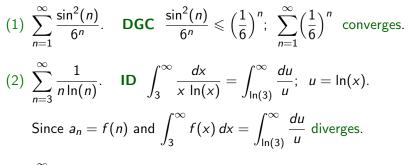
#### Example



#### Example

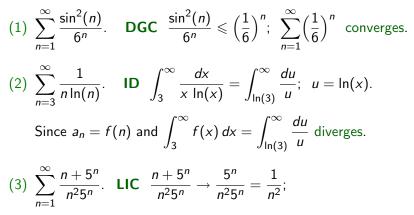


#### Example



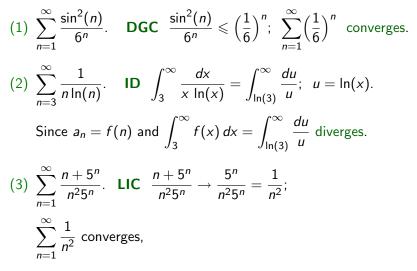
(3) 
$$\sum_{n=1}^{\infty} \frac{n+5^n}{n^2 5^n}$$
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#### Example

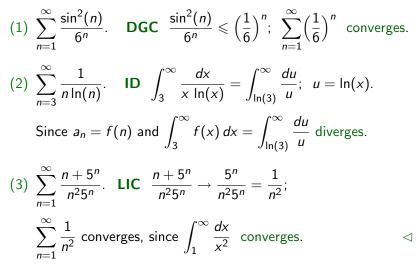


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#### Example



#### Example



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# Ratio test (Sect. 10.5)

- The ratio test.
- Using the ratio test.
- Few more examples.
- Comment: The root test.

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Theorem Let  $\{a_n\}$  be a positive sequence with  $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$  exists. (a) If  $\rho < 1$ , the series  $\sum a_n$  converges. (b) If  $\rho > 1$ , the series  $\sum a_n$  diverges. (c) If  $\rho = 1$ , the test is inconclusive.

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Theorem Let  $\{a_n\}$  be a positive sequence with  $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho$  exists. (a) If  $\rho < 1$ , the series  $\sum a_n$  converges. (b) If  $\rho > 1$ , the series  $\sum a_n$  diverges. (c) If  $\rho = 1$ , the test is inconclusive.

Remark: The ratio test compares the series  $\sum a_n$  with an appropriate geometric series  $\sum r^n$ .

**Proof:** Case (a): Since  $a_n \ge 0$ , the series  $\sum a_n$  is

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**Proof:** Case (a): Since  $a_n \ge 0$ , the series  $\sum a_n$  is non-decreasing.

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 $\frac{a_{N+n}}{2}$ 

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$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=0}^{\infty} a_{N+n}$$

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$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=0}^{\infty} a_{N+n} \leq \sum_{n=0}^{N-1} a_n + a_N \sum_{n=0}^{\infty} r^n$$

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So  $\sum_{n=0}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \frac{a_N}{1-r}$  is bounded.  
A non-decreasing, bounded above, series converges.

Proof: Case (b): Since 
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The test in inconclusive.

## Ratio test (Sect. 10.5)

- The ratio test.
- Using the ratio test.
- Few more examples.
- Comment: The root test.

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converges or not.

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Solution: We use the ratio test,

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 $\frac{a_{n+1}}{a_n}$ 

Example Determine whether the series  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converges or not.

Solution: We use the ratio test, since  $a_n = \frac{2^n}{n!} > 0$ . We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n}$$

Example Determine whether the series  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converges or not.

Solution: We use the ratio test, since  $a_n = \frac{2^n}{n!} > 0$ . We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \frac{2^n 2}{(n+1) n!} \frac{n!}{2^n}$$

Example Determine whether the series  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converges or not.

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Example Determine whether the series  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converges or not.

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Therefore,  $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ 

Example Determine whether the series  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$  converges or not.

Solution: We use the ratio test, since  $a_n = \frac{2^n}{n!} > 0$ . We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \frac{2^n 2}{(n+1) n!} \frac{n!}{2^n} = \frac{2}{(n+1)}.$$

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Therefore,  $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2}{(n+1)}$ 

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Therefore,  $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0.$ 

Since  $\rho = 0$ 

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Therefore, 
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0.$$

Since  $\rho = 0 < 1$ , the series converges.

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Example

Determine whether the series

ies 
$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$$
 converges or not.

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Example

Determine whether the series 
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 converges or not.

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Solution: We use the ratio test,

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$$
 converges or not.

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Solution: We use the ratio test, since  $a_n = \frac{(n-1)!}{(n+1)^2} > 0$ .

Example

Determine whether the series 
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 $\frac{a_{n+1}}{a_n}$ 

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$$
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$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \, \frac{(n+1)^2}{(n-1)!}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$$
 converges or not.

Solution: We use the ratio test, since  $a_n = \frac{(n-1)!}{(n+1)^2} > 0$ . Then,

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$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n(n-1)!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!}$$

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$$\frac{a_{n+1}}{a_n} = \frac{n^3 + 2n^2 + n}{n^2 + 4n + 4}$$

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### Example

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herefore, 
$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

### Example

Determine whether the series 
$$\sum_{n=1}^{\infty} rac{(n-1)!}{(n+1)^2}$$
 converges or not.

Solution: We use the ratio test, since  $a_n = \frac{(n-1)!}{(n+1)^2} > 0$ . Then,

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Therefore, 
$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} n$$

### Example

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Therefore, 
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Since  $\rho = \infty$ 

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Therefore, 
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n = \infty.$$

Since  $\rho=\infty>1$  ,

Example

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Therefore,  $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n = \infty$ .

Since  $\rho=\infty>$  1, the series diverges.

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$  converges or not.

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Determine whether the series 
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 converges or not.

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Solution: We start with the ratio test,

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$
 converges or not.  
Solution: We start with the ratio test, since  $a_n = \frac{\ln(n)}{n} \ge 0$ .

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 $\frac{a_{n+1}}{a_n}$ 



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$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)}$$

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Since  $\rho = 1$ , the ratio test is inconclusive.

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Direct comparison test:  $a_n = \frac{\ln(n)}{n}$ 

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Direct comparison test:  $a_n = \frac{\ln(n)}{n} \ge \frac{1}{n}$ 

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Since  $\rho=$  1, the ratio test is inconclusive.

Direct comparison test:  $a_n = \frac{\ln(n)}{n} \ge \frac{1}{n}$  implies that

$$\sum \frac{\ln(n)}{n} \ge \sum \frac{1}{n},$$

Example

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Direct comparison test:  $a_n = \frac{\ln(n)}{n} \ge \frac{1}{n}$  implies that

$$\sum \frac{\ln(n)}{n} \ge \sum \frac{1}{n}, \text{ which diverges.}$$

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, which diverges.

Therefore, the series diverges.

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# Ratio test (Sect. 10.5)

- The ratio test.
- Using the ratio test.
- Few more examples.
- Comment: The root test.

Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$  converges or not.

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Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$$
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Solution: We start with the ratio test,

Example

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 $\frac{a_{n+1}}{a_n}$ 

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$$
 converges or not.

Solution: We start with the ratio test, since  $a_n = \frac{\ln(7n)}{n^3} \ge 0$ .

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$$
 converges or not.

Solution: We start with the ratio test, since  $a_n = \frac{\ln(7n)}{n^3} \ge 0$ .

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Example

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Direct comparison test: Since ln(7n) < 7n, then

$$a_n = \frac{\ln(7n)}{n^3} \leqslant \frac{7n}{n^3} = \frac{7}{n^2}$$
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which converges.

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Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$$
 converges or not.

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which converges. Therefore, the series converges.

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Example

Determine whether the series  $\sum_{n=1}^{\infty} \frac{5n \ln(n)}{6^n}$  converges or not.

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 $\frac{a_{n+1}}{a_n}$ 

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Since  $\rho = \frac{1}{6} < 1$ ,

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Since  $ho=rac{1}{6}<1$ , the ratio test says that the series converges.  $\lhd$ 

# Ratio test (Sect. 10.5)

- The ratio test.
- Using the ratio test.
- Few more examples.
- Comment: The root test.

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(c) If  $\rho = 1$ , the test is inconclusive.

Theorem Let  $\{a_n\}$  be a positive sequence with  $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$  exists. (a) If  $\rho < 1$ , the series  $\sum a_n$  converges. (b) If  $\rho > 1$ , the series  $\sum a_n$  diverges.

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Remark: The root test also compares the series  $\sum a_n$  with an appropriate geometric series  $\sum r^n$ .

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So  $\sum a_n$  is bounded. A non-decreasing, bounded above, series converges. The proofs for (b), (c) are similar to ratio test.

# Alternating series and absolute convergence (Sect. 10.6)

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- Alternating series.
- Absolute and conditional convergence.
- Absolute convergence test.
- Few examples.

Definition

An infinite series  $\sum a_n$  is an *alternating series* iff holds either

$$a_n = (-1)^n |a_n|$$
 or  $a_n = (-1)^{n+1} |a_n|$ .

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Example

► The alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

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$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)n^2}{(n+1)!}$$

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Theorem (Leibniz's test)

If the sequence  $\{a_n\}$  satisfies:  $0 < a_n$ , and  $a_{n+1} \leq a_n$ , and  $a_n \to 0$ , then the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

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**Proof**: Write down the partial sum  $s_{2n}$  as follows

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - \dots + s_{2n-1} - s_{2n}$$
  
=  $(a_1 - a_2) + (a_3 - a_4) + \dots + (s_{2n-1} - s_{2n})$   
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The second expression implies  $s_{2n} \leq s_{2(n+1)}$ .

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The second expression implies  $s_{2n} \leqslant s_{2(n+1)}$ .

The third expression says that  $s_{2n}$  is bounded above.

#### Theorem (Leibniz's test)

If the sequence  $\{a_n\}$  satisfies:  $0 < a_n$ , and  $a_{n+1} \leq a_n$ , and  $a_n \to 0$ , then the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

**Proof**: Write down the partial sum  $s_{2n}$  as follows

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - \dots + s_{2n-1} - s_{2n}$$
  
=  $(a_1 - a_2) + (a_3 - a_4) + \dots + (s_{2n-1} - s_{2n})$   
=  $a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (s_{2n-2} - s_{2n-1}) - s_{2n}.$ 

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Since  $s_{2n+1} = s_{2n} + a_{2n+1}$ , and  $a_n \rightarrow 0$ , then  $s_{2n+1} \rightarrow L + 0 = L$ .

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The second expression implies  $s_{2n} \leq s_{2(n+1)}$ .

The third expression says that  $s_{2n}$  is bounded above.

Therefore converges,  $s_{2n} \rightarrow L$ .

Since  $s_{2n+1} = s_{2n} + a_{2n+1}$ , and  $a_n \to 0$ , then  $s_{2n+1} \to L + 0 = L$ . We conclude that  $\sum (-1)^{n+1} a_n$  converges.

### Example

Show that the alternating harmonic series 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$
. converges.

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Solution: Introduce the sequence  $a_n = \frac{(-1)^{n+1}}{n}$ .

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► 
$$|a_n| > 0;$$

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- ►  $|a_n| > 0;$
- ►  $|a_{n+1}| < |a_n|;$

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Show that the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ . converges.

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- |a<sub>n</sub>| > 0;
   |a<sub>n+1</sub>| < |a<sub>n</sub>|;
- ►  $|a_n| \rightarrow 0.$

### Example

Show that the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ . converges.

Solution: Introduce the sequence 
$$a_n = \frac{(-1)^{n+1}}{n}$$
.

The sequence  $\{a_n\}$  satisfies the hypothesis in the Leibniz test:

► 
$$|a_n| > 0;$$

► 
$$|a_{n+1}| < |a_n|;$$

► 
$$|a_n| \rightarrow 0.$$

We then conclude that  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges.  $\triangleleft$ 

# Alternating series and absolute convergence (Sect. 10.6)

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- Alternating series.
- ► Absolute and conditional convergence.
- Absolute convergence test.
- Few examples.

#### Remarks:

Several convergence tests apply only to positive series.

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- ► Given an arbitrary series ∑ a<sub>n</sub>, the series ∑ |a<sub>n</sub>| has non-negative terms.

### Definition

A series ∑ a<sub>n</sub> is absolutely convergent iff the series ∑ |a<sub>n</sub>| converges.

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### Remarks:

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### Definition

- A series ∑ a<sub>n</sub> is absolutely convergent iff the series ∑ |a<sub>n</sub>| converges.
- A series converges conditionally iff it converges but does not converges absolutely.

### Example

• The alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges conditionally.

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### Example

• The alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges conditionally.

Because the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges and the alternating harmonic series converges.

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### Example

► The alternating harmonic series ∑<sub>n=1</sub><sup>∞</sup> (-1)<sup>n+1</sup>/n converges conditionally.

Because the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges and the

alternating harmonic series converges.

• The geometric series 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$$
 converges absolutely.

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► The geometric series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$  converges absolutely. Because the geometric series  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  converges.

# Alternating series and absolute convergence (Sect. 10.6)

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- Alternating series.
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- Few examples.

Theorem If the series  $\sum |a_n|$  converges, then the series  $\sum a_n$  converges.

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Remark: The converse is not true.

Theorem

If the series  $\sum |a_n|$  converges, then the series  $\sum a_n$  converges.

#### Remark:

The converse is not true. A series can converge conditionally:

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$$\sum \frac{(-1)^{n+1}}{n} \text{ converges, but } \sum \left| \frac{(-1)^{n+1}}{n} \right| \text{ does not converge.}$$
Proof:  $-|a_n| \leq a_n \leq |a_n|$ 

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Proof:  $-|a_n| \leq a_n \leq |a_n| \Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|.$ 

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#### Theorem

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 $\sum \frac{(-1)^{n+1}}{n} \text{ converges, but } \sum \left| \frac{(-1)^{n+1}}{n} \right| \text{ does not converge.}$ Proof:  $-|a_n| \leq a_n \leq |a_n| \Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|.$ Since  $\sum |a_n|$  converges so does  $\sum 2|a_n|.$ 

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The converse is not true. A series can converge conditionally:

 $\sum \frac{(-1)^{n+1}}{n} \text{ converges, but } \sum \left| \frac{(-1)^{n+1}}{n} \right| \text{ does not converge.}$ Proof:  $-|a_n| \leq a_n \leq |a_n| \Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|.$ Since  $\sum |a_n|$  converges so does  $\sum 2|a_n|.$ Direct comparison test implies  $\sum (a_n + |a_n|)$  converges.

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#### Theorem

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and both series on the right-hand side converge. Hence  $\sum a_n$  converges.

# Alternating series and absolute convergence (Sect. 10.6)

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- Alternating series.
- Absolute and conditional convergence.
- Absolute convergence test.
- ► Few examples.

#### Example

Determine whether the series 
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4n}{4n^6+5}$$
 converges

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absolutely, conditionally, or does not converge at all.

#### Example

Determine whether the series 
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4n}{4n^6 + 5}$$
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Solution: We first study absolute convergence.

#### Example

Determine whether the series 
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4n}{4n^6+5}$$
 converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence  $a_n = \left| (-1)^{n+1} \frac{4n}{4n^6 + 5} \right|$ 

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Determine whether the series 
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Solution: We first study absolute convergence. We use the ratio test in the sequence  $a_n = \left| (-1)^{n+1} \frac{4n}{4n^6 + 5} \right| = \frac{4n}{4n^6 + 5}$ .

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 $\frac{a_{n+1}}{a_n}$ 

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$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]} \frac{[4n^6+5]}{4n}$$

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Ratio test inconclusive.

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$$4n^{6} < 4n^{6} + 5 \implies \frac{1}{4n^{6} + 5} < \frac{1}{4n^{6}} \implies \frac{4n}{4n^{6} + 5} < \frac{1}{n^{5}}.$$
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$$\sum_{n=1}^{\infty} \frac{1}{n^{5}} \text{ converges, (IT), so the series converges absolutely.} \triangleleft$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$$
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and ln(n) < n

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and  $\ln(n) < n$  implies  $\frac{1}{n} < \frac{1}{\ln(n)}.$   
Since the harmonic series diverges, then  $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$  diverges;

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Example

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Solution: The series diverges absolutely, since

$$\begin{split} |a_n| &= \Big| \frac{(-1)^{n+1}}{\ln(n)} \Big| = \frac{1}{\ln(n)}, \\ \text{and } \ln(n) < n \text{ implies } \frac{1}{n} < \frac{1}{\ln(n)}. \\ \text{Since the harmonic series diverges, then } \sum_{n=1}^{\infty} \frac{1}{\ln(n)} \text{ diverges;} \\ \text{therefore, the series } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)} \text{ diverges absolutely.} \end{split}$$

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Solution: Recall: The series diverges absolutely.

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#### Example

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Furthermore, the inequality  $\ln(n) < \ln(n+1)$ 

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Hence, the Leibniz test implies that 
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Hence, the series converges conditionally.

Example

Determine whether the series 
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Solution: We test absolute convergence:  $|a_n| = \left|\frac{(-100)^n}{n!}\right|$ 

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Let us check the ratio test:

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$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!} \frac{n!}{100^n}$$

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 $\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!} \frac{n!}{100^n} = \frac{100(100^n)}{(n+1) n!} \frac{n!}{100^n}$ 

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Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!} \frac{n!}{100^n} = \frac{100(100^n)}{(n+1)n!} \frac{n!}{100^n} = \frac{100}{(n+1)}$$

Example

Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$$
 converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence:  $|a_n| = \left|\frac{(-100)^n}{n!}\right| = \frac{100^n}{n!}$ .

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The ratio test implies  $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$  coverges absolutely.

Therefore, the series converges.