# Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- Direct comparison test for series.
- Review: Limit comparison test for integrals.
- Limit comparison test for series.
- Few examples.

# Comparison tests (Sect. 10.4)

- **Review: Direct comparison test for integrals.**
- Direct comparison test for series.
- Review: Limit comparison test for integrals.
- Limit comparison test for series.
- Few examples.

# Review: Direct comparison test for integrals

**Theorem (Direct comparison test)**

*If* $0 \leqslant \int_a^{\infty} f(x)\,dx \leqslant \int_a^{\infty} g(x)\,dx$, *then:*

(a) $\int_a^{\infty} g(x)\,dx$ *converges* $\Rightarrow$ $\int_a^{\infty} f(x)\,dx$ *converges;*

(b) $\int_a^{\infty} f(x)\,dx$ *diverges* $\Rightarrow$ $\int_a^{\infty} g(x)\,dx$ *diverges.*

# Review: Direct comparison test for integrals

Theorem (Direct comparison test)

*If* $0 \leqslant \int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx$, *then:*

(a) $\int_a^\infty g(x)\,dx$ *converges* $\Rightarrow$ $\int_a^\infty f(x)\,dx$ *converges;*

(b) $\int_a^\infty f(x)\,dx$ *diverges* $\Rightarrow$ $\int_a^\infty g(x)\,dx$ *diverges.*

Example

$\int_0^\infty e^{-x^2}\,dx$ converges,

# Review: Direct comparison test for integrals

Theorem (Direct comparison test)

*If* $0 \leqslant \int_a^{\infty} f(x)\,dx \leqslant \int_a^{\infty} g(x)\,dx$, *then:*

(a) $\int_a^{\infty} g(x)\,dx$ *converges* $\Rightarrow$ $\int_a^{\infty} f(x)\,dx$ *converges;*

(b) $\int_a^{\infty} f(x)\,dx$ *diverges* $\Rightarrow$ $\int_a^{\infty} g(x)\,dx$ *diverges.*

Example

$\int_0^{\infty} e^{-x^2}\,dx$ converges, since $\int_0^{\infty} e^{-x^2}\,dx \leqslant \int_0^{\infty} e^{-x}\,dx$.

# Review: Direct comparison test for integrals

Theorem (Direct comparison test)

*If* $0 \leqslant \int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx$, *then:*

(a) $\int_a^\infty g(x)\,dx$ *converges* $\Rightarrow$ $\int_a^\infty f(x)\,dx$ *converges;*

(b) $\int_a^\infty f(x)\,dx$ *diverges* $\Rightarrow$ $\int_a^\infty g(x)\,dx$ *diverges.*

Example

$\int_0^\infty e^{-x^2}\,dx$ converges, since $\int_0^\infty e^{-x^2}\,dx \leqslant \int_0^\infty e^{-x}\,dx$.

$\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges,

# Review: Direct comparison test for integrals

Theorem (Direct comparison test)

*If* $0 \leqslant \int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx$, *then:*

(a) $\int_a^\infty g(x)\,dx$ *converges* $\Rightarrow$ $\int_a^\infty f(x)\,dx$ *converges;*

(b) $\int_a^\infty f(x)\,dx$ *diverges* $\Rightarrow$ $\int_a^\infty g(x)\,dx$ *diverges.*

Example

$\int_0^\infty e^{-x^2}\,dx$ converges, since $\int_0^\infty e^{-x^2}\,dx \leqslant \int_0^\infty e^{-x}\,dx$.

$\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges, since $\int_2^\infty \frac{dx}{x} \leqslant \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$. $\triangleleft$

# Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- **Direct comparison test for series.**
- Review: Limit comparison test for integrals.
- Limit comparison test for series.
- Few examples.

# Direct comparison test for series

## Theorem

*If the sequences satisfy* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant N$, *then*

(a) $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges* $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges* $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges.*

# Direct comparison test for series

## Theorem

*If the sequences satisfy* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant N$, *then*

(a) $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges* $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges* $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges.*

## Example

Determine whether the the series $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ converges or not.

# Direct comparison test for series

## Theorem

*If the sequences satisfy* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant N$, *then*

(a) $\sum_{n=1}^{\infty} b_n$ *converges* $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\sum_{n=1}^{\infty} a_n$ *diverges* $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ *converges.*

## Example

Determine whether the the series $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ converges or not.

Solution: Since $\frac{n+2}{n^2-n} > \frac{n}{n^2-n}$

# Direct comparison test for series

## Theorem

*If the sequences satisfy* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant N$, *then*

(a) $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges* $\quad\Rightarrow\quad \displaystyle\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges* $\quad\Rightarrow\quad \displaystyle\sum_{n=1}^{\infty} b_n$ *converges.*

## Example

Determine whether the the series $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ converges or not.

Solution: Since $\dfrac{n+2}{n^2-n} > \dfrac{n}{n^2-n} = \dfrac{1}{n-1}$

# Direct comparison test for series

Theorem

*If the sequences satisfy* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant N$, *then*

(a) $\sum_{n=1}^{\infty} b_n$ *converges* $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\sum_{n=1}^{\infty} a_n$ *diverges* $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ *converges.*

Example

Determine whether the the series $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ converges or not.

Solution: Since $\frac{n+2}{n^2-n} > \frac{n}{n^2-n} = \frac{1}{n-1} > \frac{1}{n}$,

# Direct comparison test for series

## Theorem

*If the sequences satisfy* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant N$, *then*

(a) $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges* $\quad\Rightarrow\quad \displaystyle\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges* $\quad\Rightarrow\quad \displaystyle\sum_{n=1}^{\infty} b_n$ *converges.*

## Example

Determine whether the the series $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ converges or not.

Solution: Since $\dfrac{n+2}{n^2-n} > \dfrac{n}{n^2-n} = \dfrac{1}{n-1} > \dfrac{1}{n}$, we conclude that:

# Direct comparison test for series

## Theorem

*If the sequences satisfy* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant N$, *then*

(a) $\sum_{n=1}^{\infty} b_n$ *converges* $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\sum_{n=1}^{\infty} a_n$ *diverges* $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ *converges.*

## Example

Determine whether the the series $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ converges or not.

Solution: Since $\dfrac{n+2}{n^2-n} > \dfrac{n}{n^2-n} = \dfrac{1}{n-1} > \dfrac{1}{n}$, we conclude that:

$$\sum_{n=2}^{\infty} \frac{1}{n} < \sum_{n=2}^{\infty} \frac{n+2}{n^2-n}.$$

# Direct comparison test for series

## Theorem

*If the sequences satisfy* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant N$*, then*

(a) $\sum_{n=1}^{\infty} b_n$ *converges* $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\sum_{n=1}^{\infty} a_n$ *diverges* $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ *converges.*

## Example

Determine whether the the series $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ converges or not.

Solution: Since $\frac{n+2}{n^2-n} > \frac{n}{n^2-n} = \frac{1}{n-1} > \frac{1}{n}$, we conclude that:

$\sum_{n=2}^{\infty} \frac{1}{n} < \sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$. Therefore, $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges. ◁

# Direct comparison test for series

## Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

# Direct comparison test for series

Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds,

# Direct comparison test for series

Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $\;3^n \leqslant n\,3^n$

# Direct comparison test for series

## Example

Determine whether the the series $\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $3^n \leqslant n\,3^n \quad \Rightarrow \quad \frac{1}{n\,3^n} \leqslant \frac{1}{3^n}$.

# Direct comparison test for series

## Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $\;3^n \leqslant n\,3^n \quad \Rightarrow \quad \dfrac{1}{n\,3^n} \leqslant \dfrac{1}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n}$$

# Direct comparison test for series

## Example

Determine whether the the series $\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $3^n \leqslant n\,3^n \quad \Rightarrow \quad \frac{1}{n\,3^n} \leqslant \frac{1}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{3}\Big)^n$$

# Direct comparison test for series

### Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $3^n \leqslant n\,3^n \quad \Rightarrow \quad \dfrac{1}{n\,3^n} \leqslant \dfrac{1}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{3}\Big)^n = \sum_{n=0}^{\infty} \Big(\frac{1}{3}\Big)^n - 1,$$

# Direct comparison test for series

**Example**

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $\;3^n \leqslant n\,3^n \quad \Rightarrow \quad \dfrac{1}{n\,3^n} \leqslant \dfrac{1}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{3}\Big)^n = \sum_{n=0}^{\infty} \Big(\frac{1}{3}\Big)^n - 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \frac{1}{\left(1 - \frac{1}{3}\right)} - 1$$

# Direct comparison test for series

**Example**

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $\;3^n \leqslant n\,3^n \quad \Rightarrow \quad \dfrac{1}{n\,3^n} \leqslant \dfrac{1}{3^n}.$

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{3}\Big)^n = \sum_{n=0}^{\infty} \Big(\frac{1}{3}\Big)^n - 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \frac{1}{\big(1 - \frac{1}{3}\big)} - 1 = \frac{1}{\big(\frac{3-1}{3}\big)} - 1$$

# Direct comparison test for series

## Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $3^n \leqslant n\,3^n \quad \Rightarrow \quad \dfrac{1}{n\,3^n} \leqslant \dfrac{1}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{3}\Big)^n = \sum_{n=0}^{\infty} \Big(\frac{1}{3}\Big)^n - 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \frac{1}{\big(1 - \frac{1}{3}\big)} - 1 = \frac{1}{\big(\frac{3-1}{3}\big)} - 1 = \frac{3}{2} - 1$$

# Direct comparison test for series

## Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $3^n \leqslant n\,3^n \quad \Rightarrow \quad \dfrac{1}{n\,3^n} \leqslant \dfrac{1}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{3}\Big)^n = \sum_{n=0}^{\infty} \Big(\frac{1}{3}\Big)^n - 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \frac{1}{\big(1-\frac{1}{3}\big)} - 1 = \frac{1}{\big(\frac{3-1}{3}\big)} - 1 = \frac{3}{2} - 1 = \frac{1}{2}.$$

# Direct comparison test for series

## Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $3^n \leqslant n\,3^n \quad \Rightarrow \quad \dfrac{1}{n\,3^n} \leqslant \dfrac{1}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{3}\Big)^n = \sum_{n=0}^{\infty} \Big(\frac{1}{3}\Big)^n - 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \frac{1}{\left(1 - \frac{1}{3}\right)} - 1 = \frac{1}{\left(\frac{3-1}{3}\right)} - 1 = \frac{3}{2} - 1 = \frac{1}{2}.$$

We conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges. $\triangleleft$

# Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- Direct comparison test for series.
- **Review: Limit comparison test for integrals.**
- Limit comparison test for series.
- Few examples.

# Review: Limit comparison test for integrals

Theorem (Limit comparison test)

*If positive functions* $f, g : [a, \infty) \to \mathbb{R}$ *are continuous and*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad \textit{with} \quad 0 < L < \infty,$$

*then either both* $\displaystyle\int_a^\infty f(x)\,dx$, $\displaystyle\int_a^\infty g(x)\,dx$, *converge or diverge.*

## Review: Limit comparison test for integrals

### Theorem (Limit comparison test)

*If positive functions* $f, g : [a, \infty) \to \mathbb{R}$ *are continuous and*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad \textit{with} \quad 0 < L < \infty,$$

*then either both* $\displaystyle\int_a^{\infty} f(x)\,dx$, $\displaystyle\int_a^{\infty} g(x)\,dx$, *converge or diverge.*

Remark: If the integrals converge, their values may **not** agree.

# Review: Limit comparison test for integrals

## Theorem (Limit comparison test)

*If positive functions $f, g : [a, \infty) \to \mathbb{R}$ are continuous and*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad \textit{with} \quad 0 < L < \infty,$$

*then either both* $\displaystyle\int_a^\infty f(x)\,dx$, $\displaystyle\int_a^\infty g(x)\,dx$, *converge or diverge.*

Remark: If the integrals converge, their values may **not** agree.

### Example

$\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^6+1}}$ converges

# Review: Limit comparison test for integrals

### Theorem (Limit comparison test)

*If positive functions* $f, g : [a, \infty) \to \mathbb{R}$ *are continuous and*

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad \textit{with} \quad 0 < L < \infty,$$

*then either both* $\displaystyle\int_a^\infty f(x)\,dx, \ \int_a^\infty g(x)\,dx$, *converge or diverge.*

Remark: If the integrals converge, their values may **not** agree.

### Example

$\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^6 + 1}}$ converges because $\displaystyle\int_1^\infty \frac{dx}{x^3}$ converges.

# Review: Limit comparison test for integrals

### Theorem (Limit comparison test)

*If positive functions $f, g : [a, \infty) \to \mathbb{R}$ are continuous and*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad \text{with} \quad 0 < L < \infty,$$

*then either both* $\int_a^\infty f(x)\,dx$, $\int_a^\infty g(x)\,dx$, *converge or diverge.*

Remark: If the integrals converge, their values may **not** agree.

### Example

$\int_1^\infty \frac{dx}{\sqrt{x^6+1}}$ converges because $\int_1^\infty \frac{dx}{x^3}$ converges.

$\int_1^\infty \frac{dx}{\sqrt{x+\sin(x)}}$ diverges

# Review: Limit comparison test for integrals

## Theorem (Limit comparison test)

*If positive functions* $f, g : [a, \infty) \to \mathbb{R}$ *are continuous and*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad \textit{with} \quad 0 < L < \infty,$$

*then either both* $\displaystyle\int_a^\infty f(x)\,dx$, $\displaystyle\int_a^\infty g(x)\,dx$, *converge or diverge.*

Remark: If the integrals converge, their values may **not** agree.

## Example

$\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^6+1}}$ converges because $\displaystyle\int_1^\infty \frac{dx}{x^3}$ converges.

$\displaystyle\int_1^\infty \frac{dx}{\sqrt{x+\sin(x)}}$ diverges because $\displaystyle\int_1^\infty \frac{dx}{x^{1/2}}$ diverges.

# Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- Direct comparison test for series.
- Review: Limit comparison test for integrals.
- **Limit comparison test for series.**
- Few examples.

# Limit comparison test for series

**Theorem (Limit comparison test)**

*Assume that* $0 < a_n$, *and* $0 < b_n$ *for* $N \leqslant n$.

(a) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, *then the infinite series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both converge or both diverge.*

(b) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, *and* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(c) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, *and* $\sum_{n=1}^{\infty} b_n$ *diverges, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

# Limit comparison test for series

**Theorem (Limit comparison test)**

*Assume that* $0 < a_n$, *and* $0 < b_n$ *for* $N \leqslant n$.

(a) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, *then the infinite series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both converge or both diverge.*

(b) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, *and* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(c) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, *and* $\sum_{n=1}^{\infty} b_n$ *diverges, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Remark: If the series converge, their values may **not** agree.

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$ converges or not.

# Limit comparison test for series

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

# Limit comparison test for series

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2 + 7)}$$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$

# Limit comparison test for series

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$

For $n$ large $a_n = \dfrac{\sqrt{n}}{(4n^2+7)}$ behaves like

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2 + 7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2 + 7)} = \frac{\sqrt{n}}{(4n^2 + 7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4 + \left(\frac{7}{n^2}\right)}$$

For $n$ large $a_n = \dfrac{\sqrt{n}}{(4n^2 + 7)}$ behaves like $b_n = \dfrac{1}{4\, n^{3/2}}$.

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$

For $n$ large $a_n = \dfrac{\sqrt{n}}{(4n^2+7)}$ behaves like $b_n = \dfrac{1}{4\, n^{3/2}}$.

We choose $b_n = \dfrac{1}{4\, n^{3/2}}$ to do the limit comparison test.

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4 + \left(\frac{7}{n^2}\right)}$$

For $n$ large $a_n = \dfrac{\sqrt{n}}{(4n^2+7)}$ behaves like $b_n = \dfrac{1}{4\, n^{3/2}}$.

We choose $b_n = \dfrac{1}{4\, n^{3/2}}$ to do the limit comparison test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\frac{\sqrt{n}}{(4n^2+7)}\right) 4n^{3/2}$$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$

For $n$ large $a_n = \dfrac{\sqrt{n}}{(4n^2+7)}$ behaves like $b_n = \dfrac{1}{4\,n^{3/2}}$.

We choose $b_n = \dfrac{1}{4\,n^{3/2}}$ to do the limit comparison test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\frac{\sqrt{n}}{(4n^2+7)}\right) 4n^{3/2} = \lim_{n\to\infty} \frac{4n^2}{(4n^2+7)}$$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4+\left(\frac{7}{n^2}\right)}$$

For $n$ large $a_n = \dfrac{\sqrt{n}}{(4n^2+7)}$ behaves like $b_n = \dfrac{1}{4\,n^{3/2}}$.

We choose $b_n = \dfrac{1}{4\,n^{3/2}}$ to do the limit comparison test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\frac{\sqrt{n}}{(4n^2+7)}\right) 4n^{3/2} = \lim_{n\to\infty} \frac{4n^2}{(4n^2+7)} = 1.$$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ both converge or diverge.

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ both converge or diverge.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges $\Leftrightarrow$ $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}}$ converges.

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\, n^{3/2}}$ both converge or diverge.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\, n^{3/2}}$ converges $\Leftrightarrow$ $\displaystyle\int_1^{\infty} \frac{dx}{4\, x^{3/2}}$ converges.

But: $\displaystyle\int_1^{\infty} \frac{dx}{4\, x^{3/2}}$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ both converge or diverge.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges $\Leftrightarrow$ $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}}$ converges.

But: $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}} = \frac{1}{4}\,(-2)\,x^{-1/2}\Big|_1^{\infty}$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ both converge or diverge.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges $\Leftrightarrow$ $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}}$ converges.

But: $$\int_1^{\infty} \frac{dx}{4\,x^{3/2}} = \frac{1}{4}\,(-2)\,x^{-1/2}\Big|_1^{\infty} = \frac{1}{2}.$$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ both converge or diverge.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges $\Leftrightarrow$ $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}}$ converges.

But: $$\int_1^{\infty} \frac{dx}{4\,x^{3/2}} = \frac{1}{4}\,(-2)\,x^{-1/2}\Big|_1^{\infty} = \frac{1}{2}.$$

Then, the integral test says that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges.

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ both converge or diverge.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges $\Leftrightarrow$ $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}}$ converges.

But: $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}} = \frac{1}{4}\,(-2)\,x^{-1/2}\Big|_1^{\infty} = \frac{1}{2}.$

Then, the integral test says that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges.

The limit test for series says that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges. $\lhd$

# Limit comparison test for series

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

# Limit comparison test for series

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n}$$

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}}$$

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$

For $n$ large $a_n = \dfrac{3^{2n}}{(2^n + n)}$ behaves like

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$

For $n$ large $a_n = \dfrac{3^{2n}}{(2^n + n)}$ behaves like $b_n = \left(\dfrac{3}{\sqrt{2}}\right)^{2n}$.

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \left(\frac{3}{\sqrt{2}}\right)^{2n}$$

For $n$ large $a_n = \dfrac{3^{2n}}{(2^n + n)}$ behaves like $b_n = \left(\dfrac{3}{\sqrt{2}}\right)^{2n}$.

We choose $b_n = \left(\dfrac{3}{\sqrt{2}}\right)^{2n}$ to do the limit comparison test,

# Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \Big(\frac{3}{\sqrt{2}}\Big)^{2n}$$

For $n$ large $a_n = \dfrac{3^{2n}}{(2^n + n)}$ behaves like $b_n = \Big(\dfrac{3}{\sqrt{2}}\Big)^{2n}$.

We choose $b_n = \Big(\dfrac{3}{\sqrt{2}}\Big)^{2n}$ to do the limit comparison test, hence

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for $n$ large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \Big(\frac{3}{\sqrt{2}}\Big)^{2n}$$

For $n$ large $a_n = \dfrac{3^{2n}}{(2^n + n)}$ behaves like $b_n = \Big(\dfrac{3}{\sqrt{2}}\Big)^{2n}$.

We choose $b_n = \Big(\dfrac{3}{\sqrt{2}}\Big)^{2n}$ to do the limit comparison test, hence

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and both $\displaystyle\sum_{n=1}^{\infty} a_n$, $\displaystyle\sum_{n=1}^{\infty} b_n$ converge or diverge.

# Limit comparison test for series

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: Both $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$, and $\sum_{n=1}^{\infty} \Big(\frac{3}{\sqrt{2}}\Big)^{2n}$ converge or diverge.

# Limit comparison test for series

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: Both $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$, and $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ converge or diverge.

Since $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ is a geometric series

# Limit comparison test for series

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: Both $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$, and $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ converge or diverge.

Since $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ is a geometric series with ratio $r = \frac{3}{\sqrt{2}}$

# Limit comparison test for series

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: Both $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$, and $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ converge or diverge.

Since $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ is a geometric series with ratio $r = \dfrac{3}{\sqrt{2}} > 1$,

# Limit comparison test for series

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: Both $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$, and $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ converge or diverge.

Since $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ is a geometric series with ratio $r = \frac{3}{\sqrt{2}} > 1$,

the series $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$

# Limit comparison test for series

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: Both $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$, and $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ converge or diverge.

Since $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ is a geometric series with ratio $r = \frac{3}{\sqrt{2}} > 1$,

the series $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ diverges.

# Limit comparison test for series

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: Both $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$, and $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ converge or diverge.

Since $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ is a geometric series with ratio $r = \frac{3}{\sqrt{2}} > 1$,

the series $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{2}}\right)^{2n}$ diverges.

We conclude that $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ diverges. $\triangleleft$

# Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- Direct comparison test for series.
- Review: Limit comparison test for integrals.
- Limit comparison test for series.
- **Few examples.**

# Few examples

## Example

(1) $$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}.$$

# Few examples

Example

(1) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC**

# Few examples

Example

(1) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$;

# Few examples

Example

(1) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$.

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID**

# Few examples

**Example**

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x\ln(x)}$

# Few examples

Example

(1) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$. **ID** $\int_3^{\infty} \frac{dx}{x \ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$;

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n; \ \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x \ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}; \ \ u = \ln(x).$

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x\,\ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$; $u = \ln(x)$.

Since $a_n = f(n)$

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x\ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$; $u = \ln(x)$.

Since $a_n = f(n)$ and $\displaystyle\int_3^{\infty} f(x)\,dx = \int_{\ln(3)}^{\infty} \frac{du}{u}$ diverges.

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x\ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$; $u = \ln(x)$.

Since $a_n = f(n)$ and $\displaystyle\int_3^{\infty} f(x)\,dx = \int_{\ln(3)}^{\infty} \frac{du}{u}$ diverges.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5^n}{n^2 5^n}$.

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x\ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$; $u = \ln(x)$.

Since $a_n = f(n)$ and $\displaystyle\int_3^{\infty} f(x)\,dx = \int_{\ln(3)}^{\infty} \frac{du}{u}$ diverges.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n + 5^n}{n^2 5^n}$. **LIC**

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x\ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$; $u = \ln(x)$.

Since $a_n = f(n)$ and $\displaystyle\int_3^{\infty} f(x)\,dx = \int_{\ln(3)}^{\infty} \frac{du}{u}$ diverges.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5^n}{n^2 5^n}$. **LIC** $\displaystyle\frac{n+5^n}{n^2 5^n} \to \frac{5^n}{n^2 5^n} = \frac{1}{n^2}$;

# Few examples

Example

(1) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$. **ID** $\int_3^{\infty} \frac{dx}{x \ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$; $u = \ln(x)$.

Since $a_n = f(n)$ and $\int_3^{\infty} f(x)\, dx = \int_{\ln(3)}^{\infty} \frac{du}{u}$ diverges.

(3) $\sum_{n=1}^{\infty} \frac{n + 5^n}{n^2 5^n}$. **LIC** $\frac{n + 5^n}{n^2 5^n} \to \frac{5^n}{n^2 5^n} = \frac{1}{n^2}$;

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges,

# Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \left(\frac{1}{6}\right)^n$; $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x\ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$; $u = \ln(x)$.

Since $a_n = f(n)$ and $\displaystyle\int_3^{\infty} f(x)\,dx = \int_{\ln(3)}^{\infty} \frac{du}{u}$ diverges.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5^n}{n^2 5^n}$. **LIC** $\displaystyle\frac{n+5^n}{n^2 5^n} \to \frac{5^n}{n^2 5^n} = \frac{1}{n^2}$;

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, since $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$ converges. ◁

# Ratio test (Sect. 10.5)

- The ratio test.
- Using the ratio test.
- Few more examples.
- Comment: The root test.

# The ratio test

Remark: The ratio test is a way to determine whether a series converges or not.

# The ratio test

Remark: The ratio test is a way to determine whether a series converges or not.

Theorem

*Let* $\{a_n\}$ *be a positive sequence with* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ *exists.*

(a) *If* $\rho < 1$, *the series* $\sum a_n$ *converges.*

(b) *If* $\rho > 1$, *the series* $\sum a_n$ *diverges.*

(c) *If* $\rho = 1$, *the test is inconclusive.*

# The ratio test

**Remark:** The ratio test is a way to determine whether a series converges or not.

**Theorem**

*Let $\{a_n\}$ be a positive sequence with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ exists.*

(a) *If $\rho < 1$, the series $\sum a_n$ converges.*

(b) *If $\rho > 1$, the series $\sum a_n$ diverges.*

(c) *If $\rho = 1$, the test is inconclusive.*

**Remark:** The ratio test compares the series $\sum a_n$ with an appropriate geometric series $\sum r^n$.

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing.

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$,

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$,

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon$$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

$$\frac{a_{N+n}}{a_N}$$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

$$\frac{a_{N+n}}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_{N+n}}{a_{N+n-1}}$$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

$$\frac{a_{N+n}}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_{N+n}}{a_{N+n-1}} \leqslant r^n$$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

$$\frac{a_{N+n}}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_{N+n}}{a_{N+n-1}} \leqslant r^n \quad \Rightarrow \quad a_{N+n} \leqslant a_N \, r^n.$$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

$$\frac{a_{N+n}}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_{N+n}}{a_{N+n-1}} \leqslant r^n \quad \Rightarrow \quad a_{N+n} \leqslant a_N\, r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=0}^{\infty} a_{N+n}$$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

$$\frac{a_{N+n}}{a_N} = \frac{a_{N+1}}{a_N}\,\frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_{N+n}}{a_{N+n-1}} \leqslant r^n \quad \Rightarrow \quad a_{N+n} \leqslant a_N\, r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=0}^{\infty} a_{N+n} \leqslant \sum_{n=0}^{N-1} a_n + a_N \sum_{n=0}^{\infty} r^n$$

# The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

$$\frac{a_{N+n}}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_{N+n}}{a_{N+n-1}} \leqslant r^n \quad \Rightarrow \quad a_{N+n} \leqslant a_N \, r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=0}^{\infty} a_{N+n} \leqslant \sum_{n=0}^{N-1} a_n + a_N \sum_{n=0}^{\infty} r^n$$

So $\sum_{n=0}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \frac{a_N}{1-r}$ is bounded.

A non-decreasing, bounded above, series converges.

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$,

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N,$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$.

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \qquad \Rightarrow \qquad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)}$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1}$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \quad \text{diverges.}$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \quad \text{diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \quad \text{diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)^2}\right)}{\left(\frac{1}{n^2}\right)}$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \quad \text{diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \quad \text{diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1,$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \quad \text{diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1, \quad \text{converges.}$$

# The ratio test

Proof: Case (b): Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists $N$ large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots .$$

Hence, $\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \quad \text{diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1, \quad \text{converges.}$$

The test in inconclusive. □

# Ratio test (Sect. 10.5)

- The ratio test.
- **Using the ratio test.**
- Few more examples.
- Comment: The root test.

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test,

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{2^n}{n!} > 0$.

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n}$$

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n}$$

# Using the ratio test

**Example**

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \dfrac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\,\frac{n!}{2^n} = \frac{2^n\,2}{(n+1)\,n!}\,\frac{n!}{2^n}$$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \dfrac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \, \frac{n!}{2^n} = \frac{2^n\, 2}{(n+1)\, n!} \, \frac{n!}{2^n} = \frac{2}{(n+1)}.$$

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \, \frac{n!}{2^n} = \frac{2^n \, 2}{(n+1)\, n!} \, \frac{n!}{2^n} = \frac{2}{(n+1)}.$$

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

# Using the ratio test

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \dfrac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\,\frac{n!}{2^n} = \frac{2^n\,2}{(n+1)\,n!}\,\frac{n!}{2^n} = \frac{2}{(n+1)}.$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{(n+1)}$

# Using the ratio test

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \dfrac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \frac{2^n\,2}{(n+1)\,n!} \frac{n!}{2^n} = \frac{2}{(n+1)}.$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0.$

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\,\frac{n!}{2^n} = \frac{2^n\,2}{(n+1)\,n!}\,\frac{n!}{2^n} = \frac{2}{(n+1)}.$$

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0.$

Since $\rho = 0$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \dfrac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\,\frac{n!}{2^n} = \frac{2^n\,2}{(n+1)\,n!}\,\frac{n!}{2^n} = \frac{2}{(n+1)}.$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0.$

Since $\rho = 0 < 1$,

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \dfrac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\,\frac{n!}{2^n} = \frac{2^n\,2}{(n+1)\,n!}\,\frac{n!}{2^n} = \frac{2}{(n+1)}.$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0.$

Since $\rho = 0 < 1$, the series converges. $\triangleleft$

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test,

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$.

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n}$$

# Using the ratio test

**Example**

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \, \frac{(n+1)^2}{(n-1)!}$$

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!}$$

# Using the ratio test

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \, \frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2} \, \frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3 + 2n^2 + n}{n^2 + 4n + 4}$$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \, \frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2} \, \frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3 + 2n^2 + n}{n^2 + 4n + 4} = \frac{n + 2 + \frac{1}{n}}{1 + \frac{4}{n} + \frac{4}{n^2}}$$

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3 + 2n^2 + n}{n^2 + 4n + 4} = \frac{n + 2 + \frac{1}{n}}{1 + \frac{4}{n} + \frac{4}{n^2}}$$

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

# Using the ratio test

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3 + 2n^2 + n}{n^2 + 4n + 4} = \frac{n + 2 + \frac{1}{n}}{1 + \frac{4}{n} + \frac{4}{n^2}}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2} \frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3 + 2n^2 + n}{n^2 + 4n + 4} = \frac{n + 2 + \frac{1}{n}}{1 + \frac{4}{n} + \frac{4}{n^2}}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n = \infty$.

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2} \, \frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2} \, \frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3 + 2n^2 + n}{n^2 + 4n + 4} = \frac{n + 2 + \frac{1}{n}}{1 + \frac{4}{n} + \frac{4}{n^2}}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n = \infty.$

Since $\rho = \infty$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2}\,\frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2}\,\frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3+2n^2+n}{n^2+4n+4} = \frac{n+2+\frac{1}{n}}{1+\frac{4}{n}+\frac{4}{n^2}}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n = \infty.$

Since $\rho = \infty > 1$,

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2}\,\frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2}\,\frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3+2n^2+n}{n^2+4n+4} = \frac{n+2+\frac{1}{n}}{1+\frac{4}{n}+\frac{4}{n^2}}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n = \infty.$

Since $\rho = \infty > 1$, the series diverges. $\triangleleft$

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test,

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(n)}{n} \geqslant 0$.

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n}$$

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)}$$

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)}$$

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)} \to 1$$

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)} \to 1$$

Since $\rho = 1$,

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

# Using the ratio test

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: $a_n = \frac{\ln(n)}{n}$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: $a_n = \dfrac{\ln(n)}{n} \geqslant \dfrac{1}{n}$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: $a_n = \dfrac{\ln(n)}{n} \geqslant \dfrac{1}{n}$ implies that

$$\sum \frac{\ln(n)}{n} \geqslant \sum \frac{1}{n},$$

# Using the ratio test

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: $a_n = \dfrac{\ln(n)}{n} \geqslant \dfrac{1}{n}$ implies that

$$\sum \frac{\ln(n)}{n} \geqslant \sum \frac{1}{n}, \quad \text{which diverges.}$$

# Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)} \frac{n}{\ln(n)} = \frac{n}{(n+1)} \frac{\ln(n+1)}{\ln(n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: $a_n = \frac{\ln(n)}{n} \geqslant \frac{1}{n}$ implies that

$$\sum \frac{\ln(n)}{n} \geqslant \sum \frac{1}{n}, \quad \text{which diverges.}$$

Therefore, the series diverges. $\triangleleft$

# Ratio test (Sect. 10.5)

- The ratio test.
- Using the ratio test.
- **Few more examples.**
- Comment: The root test.

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test,

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(7n)}{n^3} \geqslant 0$.

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n}$$

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)}$$

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \, \frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)}$$

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0.$

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3}\,\frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$,

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

# Few more examples

**Example**

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test:

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: Since $\ln(7n) < 7n$,

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: Since $\ln(7n) < 7n$, then

$$a_n = \frac{\ln(7n)}{n^3}$$

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3}\,\frac{n^3}{\ln(7n)} = \Big(\frac{n}{n+1}\Big)^3\,\frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: Since $\ln(7n) < 7n$, then

$$a_n = \frac{\ln(7n)}{n^3} \leqslant \frac{7n}{n^3}$$

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: Since $\ln(7n) < 7n$, then

$$a_n = \frac{\ln(7n)}{n^3} \leqslant \frac{7n}{n^3} = \frac{7}{n^2}.$$

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3}\,\frac{n^3}{\ln(7n)} = \left(\frac{n}{n+1}\right)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: Since $\ln(7n) < 7n$, then

$$a_n = \frac{\ln(7n)}{n^3} \leqslant \frac{7n}{n^3} = \frac{7}{n^2}.\ \text{Hence} \sum \frac{\ln(7n)}{n^3} \leqslant \sum \frac{7}{n^2},$$

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)} = \Big(\frac{n}{n+1}\Big)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: Since $\ln(7n) < 7n$, then

$$a_n = \frac{\ln(7n)}{n^3} \leqslant \frac{7n}{n^3} = \frac{7}{n^2}.$$ Hence $\displaystyle\sum \frac{\ln(7n)}{n^3} \leqslant \sum \frac{7}{n^2}$,

which converges.

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3}\,\frac{n^3}{\ln(7n)} = \Big(\frac{n}{n+1}\Big)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: Since $\ln(7n) < 7n$, then

$a_n = \dfrac{\ln(7n)}{n^3} \leqslant \dfrac{7n}{n^3} = \dfrac{7}{n^2}$. Hence $\displaystyle\sum \frac{\ln(7n)}{n^3} \leqslant \sum \frac{7}{n^2}$,

which converges. Therefore, the series converges. $\triangleleft$

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{5n \ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test,

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{5n\ln(n)}{6^n} \geqslant 0$.

# Few more examples

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{5n\ln(n)}{6^n} \geqslant 0$.

$$\frac{a_{n+1}}{a_n}$$

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{5n\ln(n)}{6^n} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{5(n+1)\ln[(n+1)]}{6^{(n+1)}}\,\frac{6^n}{5n\ln(n)}$$

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{5n\ln(n)}{6^n} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{5(n+1)\ln[(n+1)]}{6^{(n+1)}} \frac{6^n}{5n\ln(n)}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{6}\left(\frac{n+1}{n}\right) \frac{\ln(n+1)}{\ln(n)}$$

# Few more examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{5n\ln(n)}{6^n} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{5(n+1)\ln[(n+1)]}{6^{(n+1)}} \frac{6^n}{5n\ln(n)}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{6}\left(\frac{n+1}{n}\right)\frac{\ln(n+1)}{\ln(n)} \to \frac{1}{6}$$

# Few more examples

**Example**

Determine whether the series $\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \frac{5n\ln(n)}{6^n} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{5(n+1)\ln[(n+1)]}{6^{(n+1)}} \frac{6^n}{5n\ln(n)}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{6}\left(\frac{n+1}{n}\right)\frac{\ln(n+1)}{\ln(n)} \to \frac{1}{6}$$

Since $\rho = \frac{1}{6} < 1$,

# Few more examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{5n\ln(n)}{6^n} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{5(n+1)\ln[(n+1)]}{6^{(n+1)}}\,\frac{6^n}{5n\ln(n)}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{6}\left(\frac{n+1}{n}\right)\frac{\ln(n+1)}{\ln(n)} \to \frac{1}{6}$$

Since $\rho = \dfrac{1}{6} < 1$, the ratio test says that the series converges. $\triangleleft$

# Ratio test (Sect. 10.5)

- The ratio test.
- Using the ratio test.
- Few more examples.
- **Comment: The root test.**

# Comment: The root test

**Theorem**

*Let* $\{a_n\}$ *be a positive sequence with* $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$ *exists.*

(a) *If* $\rho < 1$, *the series* $\sum a_n$ *converges.*

(b) *If* $\rho > 1$, *the series* $\sum a_n$ *diverges.*

(c) *If* $\rho = 1$, *the test is inconclusive.*

# Comment: The root test

**Theorem**

*Let $\{a_n\}$ be a positive sequence with $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$ exists.*

(a) *If $\rho < 1$, the series $\sum a_n$ converges.*

(b) *If $\rho > 1$, the series $\sum a_n$ diverges.*

(c) *If $\rho = 1$, the test is inconclusive.*

Remark: The root test also compares the series $\sum a_n$ with an appropriate geometric series $\sum r^n$.

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing.

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$,

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$,

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon$$

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N,$$

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_n \leqslant r^n.$$

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_n \leqslant r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$$

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_n \leqslant r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} r^n$$

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_n \leqslant r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} r^n$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n - \sum_{n=0}^{N-1} r^n + \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{N-1} (a_n - r^n) + \frac{1}{1-r}.$$

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_n \leqslant r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} r^n$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n - \sum_{n=0}^{N-1} r^n + \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{N-1} (a_n - r^n) + \frac{1}{1-r}.$$

So $\sum a_n$ is bounded.

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_n \leqslant r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} r^n$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n - \sum_{n=0}^{N-1} r^n + \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{N-1} (a_n - r^n) + \frac{1}{1-r}.$$

So $\sum a_n$ is bounded. A non-decreasing, bounded above, series converges.

# Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists $N$ large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_n \leqslant r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} r^n$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n - \sum_{n=0}^{N-1} r^n + \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{N-1} (a_n - r^n) + \frac{1}{1-r}.$$

So $\sum a_n$ is bounded. A non-decreasing, bounded above, series converges. The proofs for (b), (c) are similar to ratio test. □

# Alternating series and absolute convergence (Sect. 10.6)

- Alternating series.
- Absolute and conditional convergence.
- Absolute convergence test.
- Few examples.

# Alternating series

## Definition

An infinite series $\sum a_n$ is an *alternating series* iff holds either

$$a_n = (-1)^n \, |a_n| \quad \text{or} \quad a_n = (-1)^{n+1} \, |a_n|.$$

# Alternating series

## Definition

An infinite series $\sum a_n$ is an *alternating series* iff holds either

$$a_n = (-1)^n |a_n| \quad \text{or} \quad a_n = (-1)^{n+1} |a_n|.$$

## Example

- The alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

# Alternating series

## Definition

An infinite series $\sum a_n$ is an *alternating series* iff holds either

$$a_n = (-1)^n \, |a_n| \quad \text{or} \quad a_n = (-1)^{n+1} \, |a_n|.$$

## Example

- The alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

- The following series is an alternating series,

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi) n^2}{(n+1)!}$$

# Alternating series

### Definition

An infinite series $\sum a_n$ is an *alternating series* iff holds either

$$a_n = (-1)^n \, |a_n| \quad \text{or} \quad a_n = (-1)^{n+1} \, |a_n|.$$

### Example

- The alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \,.$$

- The following series is an alternating series,

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi) n^2}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \, \frac{n^2}{(n+1)!}$$

# Alternating series

### Definition

An infinite series $\sum a_n$ is an *alternating series* iff holds either

$$a_n = (-1)^n \, |a_n| \quad \text{or} \quad a_n = (-1)^{n+1} \, |a_n|.$$

### Example

- The alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

- The following series is an alternating series,

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi) n^2}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{(n+1)!} = -\frac{1}{2} + \frac{4}{6} - \frac{9}{24} + \cdots .$$

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$, *and* $a_{n+1} \leqslant a_n$, *and* $a_n \to 0$, *then the alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$, *and* $a_{n+1} \leqslant a_n$, *and* $a_n \to 0$, *then the alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof: Write down the partial sum $s_{2n}$ as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$, *and* $a_{n+1} \leqslant a_n$, *and* $a_n \to 0$, *then the alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof: Write down the partial sum $s_{2n}$ as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

The second expression implies $s_{2n} \leqslant s_{2(n+1)}$.

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$, *and* $a_{n+1} \leqslant a_n$, *and* $a_n \to 0$, *then the alternating series* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

Proof: Write down the partial sum $s_{2n}$ as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

The second expression implies $s_{2n} \leqslant s_{2(n+1)}$.

The third expression says that $s_{2n}$ is bounded above.

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$, *and* $a_{n+1} \leqslant a_n$, *and* $a_n \to 0$, *then the alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof: Write down the partial sum $s_{2n}$ as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

The second expression implies $s_{2n} \leqslant s_{2(n+1)}$.

The third expression says that $s_{2n}$ is bounded above.

Therefore converges, $s_{2n} \to L$.

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$, *and* $a_{n+1} \leqslant a_n$, *and* $a_n \to 0$, *then the alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof: Write down the partial sum $s_{2n}$ as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

The second expression implies $s_{2n} \leqslant s_{2(n+1)}$.

The third expression says that $s_{2n}$ is bounded above.

Therefore converges, $s_{2n} \to L$.

Since $s_{2n+1} = s_{2n} + a_{2n+1}$,

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$*, and* $a_{n+1} \leqslant a_n$*, and* $a_n \to 0$*, then the alternating series* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

Proof: Write down the partial sum $s_{2n}$ as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

The second expression implies $s_{2n} \leqslant s_{2(n+1)}$.

The third expression says that $s_{2n}$ is bounded above.

Therefore converges, $s_{2n} \to L$.

Since $s_{2n+1} = s_{2n} + a_{2n+1}$, and $a_n \to 0$,

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$, *and* $a_{n+1} \leqslant a_n$, *and* $a_n \to 0$, *then the alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof: Write down the partial sum $s_{2n}$ as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

The second expression implies $s_{2n} \leqslant s_{2(n+1)}$.

The third expression says that $s_{2n}$ is bounded above.

Therefore converges, $s_{2n} \to L$.

Since $s_{2n+1} = s_{2n} + a_{2n+1}$, and $a_n \to 0$, then $s_{2n+1} \to L + 0 = L$.

# Alternating series

## Theorem (Leibniz's test)

*If the sequence* $\{a_n\}$ *satisfies:* $0 < a_n$, *and* $a_{n+1} \leqslant a_n$, *and* $a_n \to 0$, *then the alternating series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof: Write down the partial sum $s_{2n}$ as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

The second expression implies $s_{2n} \leqslant s_{2(n+1)}$.

The third expression says that $s_{2n}$ is bounded above.

Therefore converges, $s_{2n} \to L$.

Since $s_{2n+1} = s_{2n} + a_{2n+1}$, and $a_n \to 0$, then $s_{2n+1} \to L + 0 = L$.

We conclude that $\sum(-1)^{n+1}a_n$ converges. □

# Alternating series

## Example

Show that the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. converges.

# Alternating series

Example

Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. converges.

Solution: Introduce the sequence $a_n = \dfrac{(-1)^{n+1}}{n}$.

# Alternating series

Example

Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. converges.

Solution: Introduce the sequence $a_n = \dfrac{(-1)^{n+1}}{n}$.

The sequence $\{a_n\}$ satisfies the hypothesis in the Leibniz test:

# Alternating series

## Example

Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. converges.

Solution: Introduce the sequence $a_n = \dfrac{(-1)^{n+1}}{n}$.

The sequence $\{a_n\}$ satisfies the hypothesis in the Leibniz test:

- $|a_n| > 0$;

# Alternating series

## Example

Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. converges.

Solution: Introduce the sequence $a_n = \dfrac{(-1)^{n+1}}{n}$.

The sequence $\{a_n\}$ satisfies the hypothesis in the Leibniz test:

- $|a_n| > 0$;
- $|a_{n+1}| < |a_n|$;

# Alternating series

## Example

Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. converges.

Solution: Introduce the sequence $a_n = \dfrac{(-1)^{n+1}}{n}$.

The sequence $\{a_n\}$ satisfies the hypothesis in the Leibniz test:

- $|a_n| > 0$;
- $|a_{n+1}| < |a_n|$;
- $|a_n| \to 0$.

# Alternating series

## Example

Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. converges.

Solution: Introduce the sequence $a_n = \dfrac{(-1)^{n+1}}{n}$.

The sequence $\{a_n\}$ satisfies the hypothesis in the Leibniz test:

- $|a_n| > 0$;
- $|a_{n+1}| < |a_n|$;
- $|a_n| \to 0$.

We then conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. $\triangleleft$

# Alternating series and absolute convergence (Sect. 10.6)

- Alternating series.
- **Absolute and conditional convergence.**
- Absolute convergence test.
- Few examples.

# Absolute and conditional convergence

## Remarks:

- Several convergence tests apply only to positive series.

# Absolute and conditional convergence

Remarks:

- Several convergence tests apply only to positive series.
- Integral test, direct comparison and limit comparison tests, ratio test, do not apply to alternating series.

# Absolute and conditional convergence

Remarks:

- Several convergence tests apply only to positive series.
- Integral test, direct comparison and limit comparison tests, ratio test, do not apply to alternating series.
- Given an arbitrary series $\sum a_n$, the series $\sum |a_n|$ has non-negative terms.

# Absolute and conditional convergence

## Remarks:

- Several convergence tests apply only to positive series.
- Integral test, direct comparison and limit comparison tests, ratio test, do not apply to alternating series.
- Given an arbitrary series $\sum a_n$, the series $\sum |a_n|$ has non-negative terms.

## Definition

- A series $\sum a_n$ is *absolutely convergent* iff the series $\sum |a_n|$ converges.

# Absolute and conditional convergence

## Remarks:

- Several convergence tests apply only to positive series.
- Integral test, direct comparison and limit comparison tests, ratio test, do not apply to alternating series.
- Given an arbitrary series $\sum a_n$, the series $\sum |a_n|$ has non-negative terms.

## Definition

- A series $\sum a_n$ is *absolutely convergent* iff the series $\sum |a_n|$ converges.
- A series *converges conditionally* iff it converges but does not converges absolutely.

# Absolute and conditional convergence

Example

- The alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

# Absolute and conditional convergence

## Example

- The alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

  Because the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the alternating harmonic series converges.

# Absolute and conditional convergence

## Example

- The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

  Because the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the alternating harmonic series converges.

- The geometric series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$ converges absolutely.

# Absolute and conditional convergence

## Example

- The alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

  Because the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the alternating harmonic series converges.

- The geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$ converges absolutely.

  Because the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

# Alternating series and absolute convergence (Sect. 10.6)

- Alternating series.
- Absolute and conditional convergence.
- **Absolute convergence test.**
- Few examples.

# Absolute convergence test

## Theorem

*If the series* $\sum |a_n|$ *converges, then the series* $\sum a_n$ *converges.*

# Absolute convergence test

**Theorem**

*If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.*

**Remark:**

The converse is not true.

# Absolute convergence test

**Theorem**

*If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.*

**Remark:**

The converse is not true. A series can converge conditionally:

# Absolute convergence test

Theorem

*If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.*

Remark:

The converse is not true. A series can converge conditionally:

$\sum \frac{(-1)^{n+1}}{n}$ converges, but $\sum \left| \frac{(-1)^{n+1}}{n} \right|$ does not converge.

# Absolute convergence test

**Theorem**

*If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.*

**Remark:**

The converse is not true. A series can converge conditionally:

$\sum \frac{(-1)^{n+1}}{n}$ converges, but $\sum \left| \frac{(-1)^{n+1}}{n} \right|$ does not converge.

**Proof:** $-|a_n| \leqslant a_n \leqslant |a_n|$

# Absolute convergence test

### Theorem

*If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.*

### Remark:

The converse is not true. A series can converge conditionally:

$$\sum \frac{(-1)^{n+1}}{n} \text{ converges, but } \sum \left| \frac{(-1)^{n+1}}{n} \right| \text{ does not converge.}$$

**Proof:** $-|a_n| \leqslant a_n \leqslant |a_n| \quad \Rightarrow \quad 0 \leqslant a_n + |a_n| \leqslant 2|a_n|.$

# Absolute convergence test

**Theorem**

*If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.*

**Remark:**

The converse is not true. A series can converge conditionally:

$$\sum \frac{(-1)^{n+1}}{n} \text{ converges, but } \sum \left|\frac{(-1)^{n+1}}{n}\right| \text{ does not converge.}$$

**Proof:** $-|a_n| \leqslant a_n \leqslant |a_n| \quad \Rightarrow \quad 0 \leqslant a_n + |a_n| \leqslant 2|a_n|$.

Since $\sum |a_n|$ converges so does $\sum 2|a_n|$.

# Absolute convergence test

**Theorem**

*If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.*

**Remark:**

The converse is not true. A series can converge conditionally:

$$\sum \frac{(-1)^{n+1}}{n} \text{ converges, but } \sum \left| \frac{(-1)^{n+1}}{n} \right| \text{ does not converge.}$$

**Proof:** $-|a_n| \leqslant a_n \leqslant |a_n| \quad \Rightarrow \quad 0 \leqslant a_n + |a_n| \leqslant 2|a_n|$.

Since $\sum |a_n|$ converges so does $\sum 2|a_n|$.

Direct comparison test implies $\sum (a_n + |a_n|)$ converges.

# Absolute convergence test

### Theorem

*If the series* $\sum |a_n|$ *converges, then the series* $\sum a_n$ *converges.*

### Remark:

The converse is not true. A series can converge conditionally:

$\sum \frac{(-1)^{n+1}}{n}$ converges, but $\sum \left|\frac{(-1)^{n+1}}{n}\right|$ does not converge.

**Proof:** $-|a_n| \leqslant a_n \leqslant |a_n| \quad \Rightarrow \quad 0 \leqslant a_n + |a_n| \leqslant 2|a_n|.$

Since $\sum |a_n|$ converges so does $\sum 2|a_n|$.

Direct comparison test implies $\sum (a_n + |a_n|)$ converges.

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|,$$

# Absolute convergence test

**Theorem**

*If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.*

**Remark:**

The converse is not true. A series can converge conditionally:

$$\sum \frac{(-1)^{n+1}}{n} \text{ converges, but } \sum \left| \frac{(-1)^{n+1}}{n} \right| \text{ does not converge.}$$

**Proof:** $-|a_n| \leqslant a_n \leqslant |a_n| \quad \Rightarrow \quad 0 \leqslant a_n + |a_n| \leqslant 2|a_n|$.

Since $\sum |a_n|$ converges so does $\sum 2|a_n|$.

Direct comparison test implies $\sum (a_n + |a_n|)$ converges.

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|,$$

and both series on the right-hand side converge.

# Absolute convergence test

**Theorem**

*If the series* $\sum |a_n|$ *converges, then the series* $\sum a_n$ *converges.*

**Remark:**

The converse is not true. A series can converge conditionally:

$\sum \frac{(-1)^{n+1}}{n}$ converges, but $\sum \left| \frac{(-1)^{n+1}}{n} \right|$ does not converge.

**Proof:** $-|a_n| \leqslant a_n \leqslant |a_n| \quad \Rightarrow \quad 0 \leqslant a_n + |a_n| \leqslant 2|a_n|$.

Since $\sum |a_n|$ converges so does $\sum 2|a_n|$.

Direct comparison test implies $\sum (a_n + |a_n|)$ converges.

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|,$$

and both series on the right-hand side converge.

Hence $\sum a_n$ converges. □

# Alternating series and absolute convergence (Sect. 10.6)

- Alternating series.
- Absolute and conditional convergence.
- Absolute convergence test.
- **Few examples.**

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence.

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right|$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n}$$

## Few examples

### Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n}$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right]$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right] \to 1.$$

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right] \to 1.$$

Ratio test inconclusive.

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right] \to 1.$$

Ratio test inconclusive. Direct comparison test:

$$4n^6 < 4n^6 + 5$$

## Few examples

### Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right] \to 1.$$

Ratio test inconclusive. Direct comparison test:

$$4n^6 < 4n^6+5 \;\Rightarrow\; \frac{1}{4n^6+5} < \frac{1}{4n^6}$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left| (-1)^{n+1} \frac{4n}{4n^6+5} \right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]} \frac{[4n^6+5]}{4n} = \frac{(n+1)}{n} \left[ \frac{4n^6+5}{4(n+1)^6+5} \right] \to 1.$$

Ratio test inconclusive. Direct comparison test:

$$4n^6 < 4n^6 + 5 \;\Rightarrow\; \frac{1}{4n^6+5} < \frac{1}{4n^6} \;\Rightarrow\; \frac{4n}{4n^6+5} < \frac{1}{n^5}.$$

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right] \to 1.$$

Ratio test inconclusive. Direct comparison test:

$$4n^6 < 4n^6+5 \;\Rightarrow\; \frac{1}{4n^6+5} < \frac{1}{4n^6} \;\Rightarrow\; \frac{4n}{4n^6+5} < \frac{1}{n^5}.$$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges,

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right] \to 1.$$

Ratio test inconclusive. Direct comparison test:

$$4n^6 < 4n^6+5 \;\Rightarrow\; \frac{1}{4n^6+5} < \frac{1}{4n^6} \;\Rightarrow\; \frac{4n}{4n^6+5} < \frac{1}{n^5}.$$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges, (IT),

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right] \to 1.$$

Ratio test inconclusive. Direct comparison test:

$$4n^6 < 4n^6+5 \;\Rightarrow\; \frac{1}{4n^6+5} < \frac{1}{4n^6} \;\Rightarrow\; \frac{4n}{4n^6+5} < \frac{1}{n^5}.$$

$\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges, (IT), so the series converges absolutely. $\triangleleft$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

# Few examples

### Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: The series diverges absolutely,

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: The series diverges absolutely, since

$$|a_n| = \left|\frac{(-1)^{n+1}}{\ln(n)}\right| = \frac{1}{\ln(n)},$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: The series diverges absolutely, since

$$|a_n| = \left|\frac{(-1)^{n+1}}{\ln(n)}\right| = \frac{1}{\ln(n)},$$

and $\ln(n) < n$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: The series diverges absolutely, since

$$|a_n| = \left| \frac{(-1)^{n+1}}{\ln(n)} \right| = \frac{1}{\ln(n)},$$

and $\ln(n) < n$ implies $\frac{1}{n} < \frac{1}{\ln(n)}$.

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: The series diverges absolutely, since

$$|a_n| = \left| \frac{(-1)^{n+1}}{\ln(n)} \right| = \frac{1}{\ln(n)},$$

and $\ln(n) < n$ implies $\frac{1}{n} < \frac{1}{\ln(n)}$.

Since the harmonic series diverges,

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: The series diverges absolutely, since

$$|a_n| = \left| \frac{(-1)^{n+1}}{\ln(n)} \right| = \frac{1}{\ln(n)},$$

and $\ln(n) < n$ implies $\frac{1}{n} < \frac{1}{\ln(n)}$.

Since the harmonic series diverges, then $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ diverges;

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: The series diverges absolutely, since

$$|a_n| = \left| \frac{(-1)^{n+1}}{\ln(n)} \right| = \frac{1}{\ln(n)},$$

and $\ln(n) < n$ implies $\frac{1}{n} < \frac{1}{\ln(n)}$.

Since the harmonic series diverges, then $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ diverges;

therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ diverges absolutely.

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

# Few examples

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0,$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0, \quad |a_n| = \frac{1}{\ln(n)} \to 0.$$

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0, \quad |a_n| = \frac{1}{\ln(n)} \to 0.$$

Furthermore, the inequality $\ln(n) < \ln(n+1)$

# Few examples

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0, \quad |a_n| = \frac{1}{\ln(n)} \to 0.$$

Furthermore, the inequality $\ln(n) < \ln(n+1)$ implies

$$|a_{n+1}| = \frac{1}{\ln(n+1)}$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0, \quad |a_n| = \frac{1}{\ln(n)} \to 0.$$

Furthermore, the inequality $\ln(n) < \ln(n+1)$ implies

$$|a_{n+1}| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)}$$

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0, \quad |a_n| = \frac{1}{\ln(n)} \to 0.$$

Furthermore, the inequality $\ln(n) < \ln(n+1)$ implies

$$|a_{n+1}| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = |a_n|.$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0, \quad |a_n| = \frac{1}{\ln(n)} \to 0.$$

Furthermore, the inequality $\ln(n) < \ln(n+1)$ implies

$$|a_{n+1}| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = |a_n|.$$

Hence, the Leibniz test implies that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges.

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0, \quad |a_n| = \frac{1}{\ln(n)} \to 0.$$

Furthermore, the inequality $\ln(n) < \ln(n+1)$ implies

$$|a_{n+1}| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = |a_n|.$$

Hence, the Leibniz test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges.

Hence, the series converges conditionally. ◁

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

# Few examples

### Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left| \frac{(-100)^n}{n!} \right|$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left|\frac{(-100)^n}{n!}\right| = \frac{100^n}{n!}$.

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left|\frac{(-100)^n}{n!}\right| = \frac{100^n}{n!}$.

Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|}$$

# Few examples

**Example**

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left|\frac{(-100)^n}{n!}\right| = \frac{100^n}{n!}$.

Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!} \frac{n!}{100^n}$$

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left|\frac{(-100)^n}{n!}\right| = \frac{100^n}{n!}$.

Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!}\,\frac{n!}{100^n} = \frac{100(100^n)}{(n+1)\,n!}\,\frac{n!}{100^n}$$

# Few examples

**Example**

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left| \dfrac{(-100)^n}{n!} \right| = \dfrac{100^n}{n!}$.

Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!} \frac{n!}{100^n} = \frac{100(100^n)}{(n+1)\, n!} \frac{n!}{100^n} = \frac{100}{(n+1)}$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left|\frac{(-100)^n}{n!}\right| = \frac{100^n}{n!}$.

Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!}\,\frac{n!}{100^n} = \frac{100(100^n)}{(n+1)\,n!}\,\frac{n!}{100^n} = \frac{100}{(n+1)} \to 0.$$

# Few examples

## Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left|\frac{(-100)^n}{n!}\right| = \frac{100^n}{n!}$.

Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!}\,\frac{n!}{100^n} = \frac{100(100^n)}{(n+1)\,n!}\,\frac{n!}{100^n} = \frac{100}{(n+1)} \to 0.$$

The ratio test implies $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ coverges absolutely.

# Few examples

## Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left|\frac{(-100)^n}{n!}\right| = \frac{100^n}{n!}$.

Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!}\,\frac{n!}{100^n} = \frac{100(100^n)}{(n+1)\,n!}\,\frac{n!}{100^n} = \frac{100}{(n+1)} \to 0.$$

The ratio test implies $\displaystyle\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ coverges absolutely.

Therefore, the series converges. $\lhd$