Review for Exam 3.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers: 8.3, 8.4, 7.5, 8.7, 10.1.
  - Trigonometric substitutions (8.3).
  - Integration using partial fractions (8.4).
  - L'Hôpital's rule (7.5).
  - Improper integrals (8.7).
  - Infinite sequences (10.1).
- Section not covered:
  - Integration using tables (8.5).

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  - Integration using tables (8.5).

Trigonometric substitutions (8.3)

Recall: From Sect. 8.2: $\int \sec(\theta)\, d\theta = \ln\big(\sec(\theta) + \tan(\theta)\big) + c.$

Example

Evaluate $I = \int \frac{e^x\, dx}{\sqrt{e^{2x} + 9}}$. Express you result in terms of the variable x.

Trigonometric substitutions (8.3)

Recall: From Sect. 8.2: $\int \sec(\theta)\, d\theta = \ln\big(\sec(\theta) + \tan(\theta)\big) + c$.

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Solution: First substitution,

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Evaluate $I = \int \frac{e^x\, dx}{\sqrt{e^{2x} + 9}}$. Express you result in terms of the variable x.

Solution: First substitution, $y = e^x$,

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Evaluate $I = \int \frac{e^x\,dx}{\sqrt{e^{2x} + 9}}$. Express you result in terms of the variable x.

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Example

Evaluate $I = \int \frac{e^x\, dx}{\sqrt{e^{2x} + 9}}$. Express you result in terms of the variable x.

Solution: First substitution, $y = e^x$, then $dy = e^x\, dx$, $y > 0$,

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Recall: From Sect. 8.2: $\int \sec(\theta)\,d\theta = \ln\big(\sec(\theta) + \tan(\theta)\big) + c.$

Example

Evaluate $I = \int \frac{e^x\,dx}{\sqrt{e^{2x}+9}}$. Express you result in terms of the variable x.

Solution: First substitution, $y = e^x$, then $dy = e^x\,dx$, $y > 0$,

$$I = \int \frac{dy}{\sqrt{y^2+9}}.\quad \text{Second subs.:}$$

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Recall: From Sect. 8.2: $\int \sec(\theta)\, d\theta = \ln\big(\sec(\theta) + \tan(\theta)\big) + c.$

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$$I = \int \frac{dy}{\sqrt{y^2 + 9}}. \quad \text{Second subs.:} \quad \begin{cases} y = 3\tan(\theta), \\ dy = 3\, \sec^2(\theta)\, d\theta, \\ \theta \in (0, \pi/2). \end{cases}$$

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$$I = \int \frac{3\sec^2(\theta)\,d\theta}{\sqrt{9\tan^2(\theta) + 9}}$$

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Example

Evaluate $I = \int \frac{e^x \, dx}{\sqrt{e^{2x} + 9}}$. Express you result in terms of the variable x.

Solution: So: $I = \int \frac{\sec^2(\theta)\, d\theta}{|\sec(\theta)|}$; $e^x = y = 3\tan(\theta)$; $\theta \in \left(0, \frac{\pi}{2}\right)$.

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Recall, $\tan(\theta) = \frac{e^x}{3}$, hence $\sec(\theta) = \sqrt{\tan^2(\theta) + 1} = \sqrt{\frac{e^{2x}}{9} + 1}$.

We conclude that,

$$I = \ln\left(e^x + \frac{1}{3}\sqrt{e^{2x} + 9}\right) + c.$$

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- **Integration using partial fractions (8.4).**
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Recall: If the polynomial in the numerator has larger degree than the polynomial in the denominator, then do the long division first.

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$$x_{\pm} = \frac{1}{2}\big(1 \pm \sqrt{1+24}\big)$$

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$$\frac{(x-2)}{(x-3)(x+2)} = \frac{a}{(x-3)} + \frac{b}{(x+2)}$$

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Integration using partial fractions (8.4)

Example

Evaluate $I = \displaystyle\int \frac{(x-2)}{(x^2-x-6)}\,dx$.

Solution: Recall that:

$$I = \int \left[\frac{a}{(x-3)} + \frac{b}{(x+2)}\right] dx; \quad x - 2 = a(x+2) + b(x-3).$$

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Evaluating at $x = 3$

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Evaluating at $x = 3$ we get $a = \frac{1}{5}$,

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Integration using partial fractions (8.4)

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Evaluating at $x = 3$ we get $a = \frac{1}{5}$, and at $x = -2$ we get $b = \frac{4}{5}$.

$$I = \frac{1}{5} \int \left[\frac{1}{(x-3)} + \frac{4}{(x+2)}\right] dx$$

Integration using partial fractions (8.4)

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Evaluate $I = \int \frac{(x-2)}{(x^2 - x - 6)}\, dx$.

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Evaluating at $x = 3$ we get $a = \frac{1}{5}$, and at $x = -2$ we get $b = \frac{4}{5}$.

$$I = \frac{1}{5} \int \left[\frac{1}{(x-3)} + \frac{4}{(x+2)} \right] dx = \frac{1}{5}\big(\ln|x-3| + 4 \ln|x+2| \big) + c.$$

Integration using partial fractions (8.4)

Example

Evaluate $I = \displaystyle\int \frac{(x-2)}{(x^2 - x - 6)}\,dx.$

Solution: Recall that:

$$I = \int \Big[\frac{a}{(x-3)} + \frac{b}{(x+2)}\Big]\,dx; \quad x - 2 = a(x+2) + b(x-3).$$

Evaluating at $x = 3$ we get $a = \dfrac{1}{5}$, and at $x = -2$ we get $b = \dfrac{4}{5}$.

$$I = \frac{1}{5}\int \Big[\frac{1}{(x-3)} + \frac{4}{(x+2)}\Big]\,dx = \frac{1}{5}\big(\ln|x-3| + 4\ln|x+2|\big) + c.$$

We conclude that $I = \ln\big(|x-3|^{1/5}(x+2)^{4/5}\big) + c.$ $\triangleleft$

Integration using partial fractions (8.4)

Remark: Incomplete summary of partial fraction decompositions:

- $$\frac{p_2(x)}{(x - r_1)(x - r_2)(x - r_3)} = \frac{c_1}{(x - r_1)} + \frac{c_2}{(x - r_2)} + \frac{c_3}{(x - r_3)}.$$

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- $$\frac{p_5(x)}{(x^2+b^2)^3} = \frac{(c_1x+c_2)}{(x^2+b^2)} + \frac{(c_3x+c_4)}{(x^2+b^2)^2} + \frac{(c_5x+c_6)}{(x^2+b^2)^3}.$$
- $$\frac{p_4(x)}{(x-r_1)(x^2+b^2)^2} = \frac{c_1}{(x-r_1)} + \frac{(c_2x+c_3)}{(x^2+b^2)} + \frac{(c_4x+c_5)}{(x^2+b^2)^2}.$$

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- Integration using partial fractions (8.4).
- **L'Hôpital's rule (7.5).**
- Improper integrals (8.7).
- Infinite sequences (10.1).

L'Hôpital's rule (7.5)

Example

Evaluate the limit $L = \lim_{x\to\infty} \left(1 - \dfrac{2}{x} - \dfrac{3}{x^2}\right)^{8x}$.

L'Hôpital's rule (7.5)

Example

Evaluate the limit $L = \lim_{x \to \infty} \left(1 - \dfrac{2}{x} - \dfrac{3}{x^2}\right)^{8x}$.

Solution: We first rewrite the limit as follows,

$$L = \lim_{x \to \infty} \left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{8x}$$

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Solution: We first rewrite the limit as follows,

$$L = \lim_{x\to\infty} \left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{8x} = \lim_{x\to\infty} e^{\left[8x\,\ln\left(1-\frac{2}{x}-\frac{3}{x^2}\right)\right]}$$

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Example

Evaluate the limit $L = \lim_{x\to\infty} \left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{8x}$.

Solution: We first rewrite the limit as follows,

$$L = \lim_{x\to\infty} \left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{8x} = \lim_{x\to\infty} e^{\left[8x \ln\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)\right]}$$

$$L = e^{\lim_{x\to\infty}\left[8x \ln\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)\right]}$$

L'Hôpital's rule (7.5)

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Evaluate the limit $L = \lim_{x\to\infty}\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{8x}$.

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$$\lim_{x\to\infty} \frac{\ln\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)}{\frac{1}{8x}} = \lim_{x\to\infty} \frac{\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{-1}\left(\frac{2}{x^2} + \frac{6}{x^3}\right)}{\left(-\frac{1}{8x^2}\right)}$$

L'Hôpital's rule (7.5)

Example

Evaluate the limit $L = \lim_{x\to\infty}\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{8x}$.

Solution: Recall: $L = e^{\lim_{x\to\infty}\left[\frac{\ln\left(1-\frac{2}{x}-\frac{3}{x^2}\right)}{\frac{1}{8x}}\right]}$, and

$$\tilde{L} = \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}-\frac{3}{x^2}\right)}{\frac{1}{8x}} = \lim_{x\to\infty}\frac{\left(1-\frac{2}{x}-\frac{3}{x^2}\right)^{-1}\left(\frac{2}{x^2}+\frac{6}{x^3}\right)}{\left(-\frac{1}{8x^2}\right)}$$

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$$\tilde{L} = \lim_{x\to\infty} -8\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{-1}\left(2 + \frac{6}{x}\right)$$

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We conclude that $L = e^{-16}$. $\triangleleft$

Review for Exam 3.

- Trigonometric substitutions (8.3).
- Integration using partial fractions (8.4).
- L'Hôpital's rule (7.5).
- **Improper integrals (8.7).**
- Infinite sequences (10.1).

Improper integrals (8.7)

Example

Evaluate the integral $I = \int_0^5 \frac{(x+1)}{\sqrt{25 - x^2}}\, dx.$

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$$I_1 = \int_0^5 \frac{x}{\sqrt{25 - x^2}}\, dx$$

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$$I_1 = \int_0^5 \frac{x}{\sqrt{25-x^2}}\,dx = \int_{25}^0 -\frac{1}{\sqrt{u}}\,\frac{du}{2}$$

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$$I_1 = \int_0^5 \frac{x}{\sqrt{25-x^2}}\,dx = \int_{25}^0 -\frac{1}{\sqrt{u}}\,\frac{du}{2} = \frac{1}{2}\int_0^{25} u^{-1/2}\,du.$$

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Example

Evaluate the integral $I = \int_0^5 \frac{(x+1)}{\sqrt{25-x^2}}\,dx$.

Solution: Recall: $I_1 = \int_0^5 \frac{x}{\sqrt{25-x^2}}\,dx = 5$.

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In the second integral: $x = 5\sin(\theta)$,

Improper integrals (8.7)

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Improper integrals (8.7)

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Evaluate the integral $I = \displaystyle\int_0^5 \frac{(x+1)}{\sqrt{25-x^2}}\,dx.$

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We conclude that $I = 5 + \frac{\pi}{2}$. $\triangleleft$

Improper integrals (8.7): Comparison tests

- Direct comparison test:

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  If $0 \leqslant f(x) \leqslant g(x)$ for $x \in [a, \infty)$, then holds

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  (a) $\int_a^{\infty} g(x)\, dx$ converges $\Rightarrow$ $\int_a^{\infty} f(x)\, dx$ converges;

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  $$\int_a^\infty f(x)\,dx, \qquad \int_a^\infty g(x)\,dx$$

  both converge or both diverge.

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Therefore, we use the limit comparison test with $g(x) = x^{-3/2}$.

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Then, by construction,

$$\lim_{x\to\infty} \left(\frac{x}{\sqrt{x^5 + x^3}}\right)\left(\frac{1}{x^{-3/2}}\right)$$

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Therefore, we use the limit comparison test with $g(x) = x^{-3/2}$.
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Improper integrals (8.7): Comparison tests

Example

Determine whether $I = \int_3^{\infty} \frac{x\,dx}{\sqrt{x^5 + x^3}}$ converges or not.

Solution: First, find an appropriate function $g(x)$ such that:

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we conclude that I converges. ◁

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We conclude that I converges. ◁

Review for Exam 3.

- Trigonometric substitutions (8.3).
- Integration using partial fractions (8.4).
- L'Hôpital's rule (7.5).
- Improper integrals (8.7).
- **Infinite sequences (10.1).**

Infinite sequences (10.1)

Example

Evaluate $L = \lim_{n\to\infty} \left(\frac{8}{3n}\right)^{\frac{1}{3n}}$.

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Evaluate $L = \lim_{n\to\infty} \left(\frac{8}{3n}\right)^{\frac{1}{3n}}$.

Solution: We study a similar limit for the function $f(x) = \left(\frac{8}{3x}\right)^{\frac{1}{3x}}$.

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Now, L'Hôpital's rule to find the limit in the exponent;

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Hence, $\lim_{x\to\infty} f(x) = e^0 = 1$, therefore, $\lim_{n\to\infty} \left(\frac{8}{3n}\right)^{\frac{1}{3n}} = 1$. $\triangleleft$

Infinite series (Sect. 10.2)

- Series and partial sums.
- Geometric series.
- The *n*-term test for a divergent series.
- Operations with series.
- Adding-deleting terms and re-indexing.

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$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

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The sequence $\left\{a_n = \frac{1}{2^n}\right\}_{n=1}^{\infty}$ defines the series

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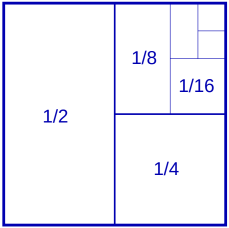
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

This infinite sum makes sense, since



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$$\vdots$$

The series converges to L iff the sequence of partial sums $\{s_n\}$ converges to L, and in this case we write $\sum_{n=1}^{\infty} a_n = L$. The series diverges iff the sequence of partial sums $\{s_n\}$ diverges.

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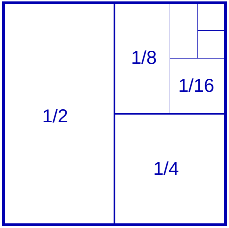

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We conclude: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$ $\triangleleft$

Infinite series (Sect. 10.2)

- Series and partial sums.
- **Geometric series.**
- The n-term test for a divergent series.
- Operations with series.
- Adding-deleting terms and re-indexing.

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Definition

A geometric series is a series of the form

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where a and r are real numbers.

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Theorem

If the geometric series $\sum_{n=0}^{\infty} a\, r^n$ *has ratio* $|r| < 1$, *then converges,*

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Since $|r| < 1$, then $r^{n+1} \to 0$. $\square$

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Evaluate the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$.

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Evaluate the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$.

Solution: Recall the picture says $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

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$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\left(\frac{2-1}{2}\right)} \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2. \qquad \triangleleft$$

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Infinite series (Sect. 10.2)

- Series and partial sums.
- Geometric series.
- **The n-term test for a divergent series.**
- Operations with series.
- Adding-deleting terms and re-indexing.

The *n*-term test for a divergent series

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$.

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Infinite series (Sect. 10.2)

- Series and partial sums.
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Operations with series

Remark: Additions of convergent series define convergent series.

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Theorem

If the series $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$, *then*

- $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$;
- $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$;
- $\sum_{n=1}^{\infty} k a_n = kA$.

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- Series and partial sums.
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The integral test (Sect. 10.3)

- Review: Bounded and monotonic sequences.
- Application: The harmonic series.
- Testing with an integral.
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A sequence $\{a_n\}$ is bounded above iff there is $M \in \mathbb{R}$ such that

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Determine whether the sequence $a_n = \dfrac{n}{n^2 + 1}$ converges or not.

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We conclude that a_n converges. $\triangleleft$

The integral test (Sect. 10.3)

- Review: Bounded and monotonic sequences.
- **Application: The harmonic series.**
- Testing with an integral.
- Error estimation in the integral test.

Application: The harmonic series

Remarks:

- The partial sums of the harmonic series, $s_n = \sum_{k=1}^{n} \frac{1}{k}$,

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- We now show that $\{s_n\}$ is unbounded from above.

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Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

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Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution: Notice the following inequalities:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] + \cdots$$

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Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution: Notice the following inequalities:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} > 1 + \frac{1}{2} + \left[\frac{2}{4}\right] + \left[\frac{4}{8}\right] + \cdots$$

Application: The harmonic series

Remarks:

- The partial sums of the harmonic series, $s_n = \sum_{k=1}^{n} \frac{1}{k}$, define an increasing sequence: $s_{n+1} > s_n$.
- We now show that $\{s_n\}$ is unbounded from above.

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The integral test (Sect. 10.3)

- Review: Bounded and monotonic sequences.
- Application: The harmonic series.
- **Testing with an integral.**
- Error estimation in the integral test.

Testing with an integral

Remark:

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Theorem

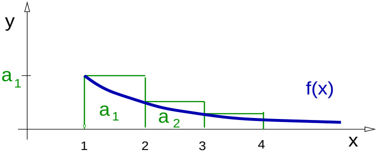
If $f : [1, \infty) \to \mathbb{R}$ is a continuous, positive, decreasing function, and $a_n = f(n)$, then the following holds:

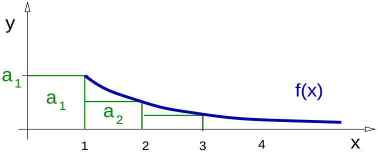
$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \Leftrightarrow \quad \int_1^{\infty} f(x)\, dx \quad \text{converges.}$$

Testing with an integral

Testing with an integral

Proof: Recall: $a_n = f(n)$. The proof is based in the pictures:





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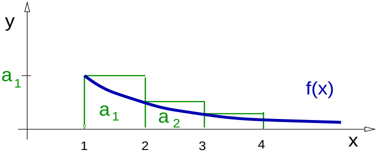

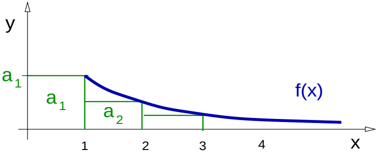

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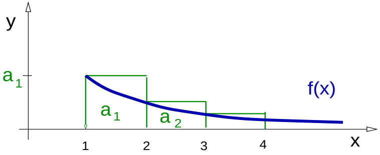

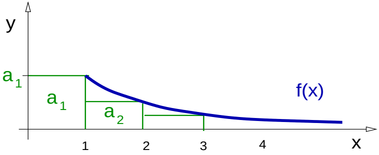

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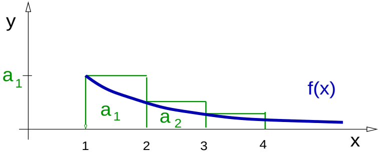

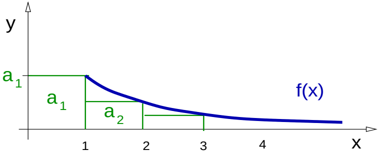

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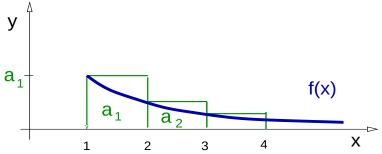

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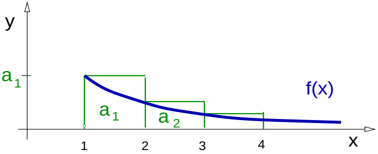

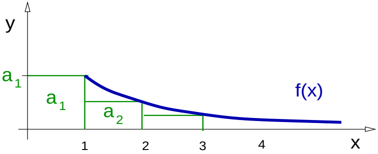

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$$\int_1^{n+1} f(x)\,dx \leqslant \sum_{k=1}^{n} a_n \leqslant a_1 + \int_1^n f(x)\,dx. \qquad \square$$

Testing with an integral

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Testing with an integral

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Limit test for improper integrals:

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Integral test for series implies: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\sqrt{n+1}}$ diverges. $\triangleleft$

The integral test (Sect. 10.3)

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Error estimation in the integral test.

Theorem

If $f : [1, \infty) \to \mathbb{R}$ *is a continuous, positive, decreasing function, and the series* $\sum_{k=1}^{n} a_k = s_n \to S$, *where* $a_n = f(n)$, *then the remainder* $R_n = S - s_n$ *satisfies*

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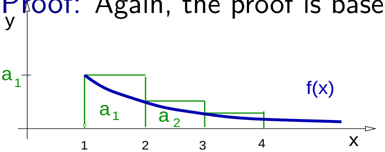

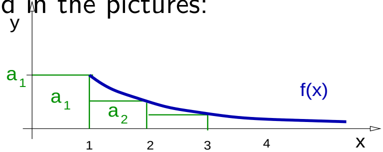

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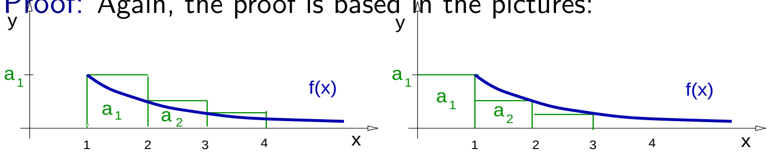

$$\int_{3}^{\infty} f(x)\,dx \leqslant R_2 = a_3 + a_4 + \cdots \leqslant \int_{2}^{\infty} f(x)\,dx$$

□